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RESEARCH ARTICLE

Advancing the concept of triangulation from social sciences research to mathematics education

Sergei Abramovich

School of Education and Professional Studies, State University of New York at Potsdam, Potsdam, NY 13676, USA

Correspondence to: Sergei Abramovich, Department of Elementary Education, School of Education and Professional Studies, State University of New York at Potsdam, Potsdam, NY 13676, USA;
E-mail: abramovs@potsdam.edu

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Abstract: The paper suggests interpreting the term triangulation, commonly used in social science research, as multiple ways of solving a problem in the context of mathematics education. The availability of different technological tools provides new perspectives on problem solving as modeling from where ideas for problem posing stem. Using topics from geometry and trigonometry, triangulation is considered through lens of teacher education. Reflections by teacher candidates on activities which are shared and reviewed in the paper indicate future teachers' readiness to implement the pedagogy of triangulated perspectives on problem solving and posing in their own mathematics classrooms.

Keywords: triangulation, mathematics education, teacher education, computational thinking, geometry, trigonometry

1 Introduction

Triangle is the basic geometric figure the properties of which have been used in real-life applications since ancient times. Its shape appears in the Egyptian Papyrus Roll (1650 BC), known as the Rhind mathematical papyrus [1], resembling an isosceles triangle the height of which is visually indistinguishable from a lateral side. This type of triangle allowed ancient Egyptians to calculate its area by replacing isosceles triangle by rectangle [2]. In turn, the genesis of emphasizing isosceles triangle in the papyrus goes back to the architecture of Neolithic age (10, 200 BC – 2000 BC) when houses had roofs in the form of an isosceles triangle with lateral sides almost touching the ground [3]. Thousand years after the Egyptians, a Greek mathematician Thales used similar and right triangles to measure indirectly large heights (*e.g.*, of pyramids) and long distances (*e.g.*, in seas).

This brief historical introduction makes it clear that the word triangulation, whatever its meaning, has mathematical origin. Whereas triangulation in the social science research is a relatively new concept aimed at improving the validity of findings based on alternative epistemologies [4], the idea of validation of mathematical propositions and/or simplification of their proofs using alternative reasoning techniques, going back to antiquity [5], is a commonplace of pure mathematics research (*e.g.*, Grinshpan (1999) [6]). In Löwe & van Kerkhove (2019) [7], the term triangulation is used to discuss from a philosophical perspective the issue of confidence of research mathematicians in the correctness of proof of a mathematical proposition. Indeed, "the correctness of mainstream mathematical proof is almost never established by formal means, but rather by informal discussion between mathematicians and peer review of papers" [8] (pp. 1398, 1399). In Sharma (2013) [9], the concept of triangulation was considered when pointing at the limitations of the interview-based qualitative methods in mathematics education research. Similarly, in Bachman et al. (2020) [10] the term triangulation refers to the collection of a variety of qualitative data regarding parental influence on the development of mathematical skills in the early childhood.

Consistent with a view that "the term 'triangulation' is used in many different ways" [11] (p. 215), one usage of the term (introduced in this paper) may be associated with mathematics education to be understood as using multiple ways of solving a problem, often enabling a problem solver, from a pure pragmatic perspective, to "check the result" [12] (p. 59). Sometimes, alternative problem-solving strategies produce different forms of answers (*e.g.*, in trigonometric equations) leading to "triangulation *between* methods" [11] (p. 215) through posing and solving new problems that deal with comparison and proving numerical equivalence of those different forms. In order to deal with alternative problem-solving strategies, a problem solver must understand mathematics behind the concepts involved.

As mentioned elsewhere [13], one can distinguish between two levels of conceptual understanding used in problem solving – basic conceptual understanding (BCU) and advanced conceptual understanding (ACU). The purpose of using BCU is to activate problem solving. ACU can serve at least two purposes: complete problem solving by advancing BCU and pose a new problem by reflecting on the one already solved. As will be shown in this paper, the process of triangulation in mathematics education and within its teacher education component, involves continuous interaction between problem solving and problem posing. Through this process, ACU at one level can be used as a BCU at another, higher level of mathematical thinking, which then leads to a new ACU. In what follows, it is assumed that the appropriate combination of BCU and ACU is required in order to carry out triangulation as multiple ways of solving a problem that converge to equivalent representations of the results to the extent of their symbolic forms (see Section 4).

According to Denzin [14], whereas a non-linear history of the term triangulation suggests that its nature is "unsettling and unruly" (p. 5079), the concept of triangulation was to help qualitative social science researchers to become more rigorous. Some mathematics teachers, with conservative (often resistant to be changed) beliefs developed during their own learning of the subject matter [15–17], might suggest that solving a problem in more than one way causes anxiety and confusion among students, thereby diminishing teachers' control of the classroom. Nonetheless, triangulation in mathematics education provides more rigor to the process of problem solving by connecting different concepts, techniques and, in the digital era, computational experiments. The credibility of results in mathematical problem solving with experimental (not necessarily digital) assistance was described by Freudenthal [18] as follows: "It is independency of new experiments that enhances credibility … [for] repeating does not create new evidence, which in fact is successfully aspired to by independent experiments" (pp. 193–194). Nowadays, whereas "engineering artifacts are unfailingly reliable ... in the realm of computers, unreliability sometimes seems to be the norm" [8] (p. 1400). As will be shown in Section 4, it is the reality of this 'norm' that calls for triangulation in the use of more than one computer program when checking the results of problem solving.

Furthermore, when symbolic computations are outsourced to software, the accuracy of the result is hanging on the accuracy of an algorithm involved, including the accuracy of the programming code used. For example, typing in the context of *Maple* the product "(n+1)(n+2)" expecting it to be expanded yields $n(n+2)+1$, which, however, is not correct as $(n+1)(n+2) = n^2+3n+2$. The problem here is in the missing multiplication sign. Yet, typing "(n+1)*(n+2)" with the asterisk between two factors as the legit multiplication sign is not enough as it does not produce the desired result either. What is needed is to follow with the command "expand(%)" (alternatively, "collect(%, n)") which would yield the correct result (the percentage symbol in the *Maple* language means "the latter"). At the same time, typing in the input box of *Wolfram Alpha* "(n+1)(n+2)" does provide the correct result, yet along with information that might confuse users if they do not understand what kind of result one should expect following a simple algebraic multiplication of two binomials. This issue was recognized in [19] by noting that in the digital age, with an easy access to the Internet, one has to learn how to manage the abundance of information provided. Indeed, in the words of one teacher candidate, "*with the use of technology, students have much more information that is accessible and are constantly learning and evolving. With this comes a greater diversity of problem-solving.*" The candidate implicitly points at the variety of information the Internet provides and is explicit in their belief that the diversity of problem solving is positively affected by students' use of technology. However, even solving simple mathematical problems with technology requires certain level of mathematical sophistication and computational thinking [20] in order to validate the result across several digital instruments as a multiple triangulation. Furthermore, when using technology in problem solving one should begin to think about a problem in hand with some initial understanding of how its solution might look like [21]. This kind of understanding may be developed through interaction with peers and the teacher.

This paper intends to demonstrate how in the digital era, promoting the concept of triangulation in K-12 mathematics teacher education as using more than one problem-solving technique/method for a single problem makes it possible to uplift epistemic development of teacher candidates in the subject matter and boost their practical competence in the use of various technologies. To this end, two mathematical illustrations from different grade levels – a problem about perimeter of an integer-sided *triangle* (Problem 1) and a problem of solving a *trigonometric* equation (Problem 2) – will be considered. Note that because trigonometry is the study of relationships among the elements of a triangle, both problems are directly related to the term triangulation. Nonetheless, the concept of triangulation understood as using multiple ways of solving a problem is applicable to any branch of school mathematics including arithmetic, algebra, geometry, combinatorics, probability and statistical data analysis. The paper is a reflection on the author's work over the years with different populations of the candidates in technology-enhanced courses emphasizing epistemic and pragmatic values of pedagogy of providing more than one correct answer through imparting alternative solutions to a mathematical problem [22].

2 Methods

The content of this paper stems from the author's work with K-12 mathematics teacher candidates in technology-rich undergraduate and graduate courses. The courses were designed for the candidates by using methods the goal of which is to "develop deep understanding of mathematics they will teach ... [through] a sustained immersion in mathematics that includes performing experiments and grappling with problems" [23] (pp. 7, 65). Achieving this goal would ultimately support the vision of a mathematical classroom where "students check their answers to problems using a different method ... understand the approaches of others to solving complex problems and identify correspondences between different approaches" [24] (p. 6). The courses have been taught over more than two decades that, in particular, generated numerous reflections of the candidates on the ideas which are referred to in this paper as triangulation in mathematical problems solving and on the value of using these ideas in their own teaching – past, present, and future. Some of those reflections are shared throughout the paper.

Materials and methods specific for mathematics education used in this paper include computer-based mathematics, standards-based mathematics education and its connection to the concept of triangulation in the social sciences research. The first type of materials consists of Excel spreadsheet – a computer program included in the Microsoft Office Package; *Wolfram Alpha* – computational knowledge engine available free online, *Maple* – mathematical software for STEM subjects and research; *The Graphing Calculator* – software capable of graphing relations from any two-variable equations and inequalities; and *The Geometer's Sketchpad* – a dynamic geometry application used in this paper for the construction of right triangles in the context of trigonometry.

The second type of materials includes K-12 mathematics teaching and learning standards from five continents. The standards uniformly call for the teaching of multiple solution strategies in solving a single problem, for problem solving using technology, for encouraging students to ask questions while expecting teachers not to reject a challenge coming from students' questions, to make mathematical connections through multiple representations, to pose new problems, and to foster computational thinking [20]. The author's accentuation of the worldwide standards is due to the fact that the university where the author prepares teacher candidates to teach mathematics is located in upstate New York in close proximity to Canada, and many of the author's students have been Canadians pursuing their master's degrees in education. This diversity of students suggests the importance of aligning mathematics education courses with multiple international perspectives on teaching and learning K-12 mathematics. This is another type of triangulation in mathematics education – verifying the acceptance of a teaching approach across various international perspectives on mathematics education. It appears that regardless of the direction a country takes in the teaching of mathematics, the focus on triangulation in mathematics education as using multiple solution strategies in problem solving enables classroom teachers around the world "to give full attention to alternative possibilities" [25] (p. 30) which are provided by innate diversity and intrinsic connectivity of mathematical methods.

The third type of materials used in this paper includes articles, book chapters and monographs on triangulation as a concept used by social science researchers since the mid 20$^{\text{th}}$ century [4, 11, 14, 26–30]. Connecting the concepts of social sciences and mathematics education intends to demonstrate that while the need for rigor was one of the main reasons for introducing the concept of triangulation into the former disciplines' research, the rigor is necessary for the success of learning mathematics and understanding its concepts in the age of technology both in contextual and decontextual situations. It is the appreciation of rigor in general that unites the ideas of triangulation across different social science and STEM disciplines.

3 Multiple methods of solution to a simple problem

In the modern-day mathematics education, multiple forms of validity of methods, stemming from different epistemologies, advocated in Denzin (2007) and Saukko (2003) [14, 30] can be interpreted in didactical terms as inquiry into which method of solution – symbolic, visual, or experimental – is the most applicable to a given problem. Often, such inquiry can be described as triangulation *within* a method [11]. Just as accepting the value of multiple forms of validity in the contemporary social sciences research, the value of multiple ways of solving a problem is not only accepted but strongly encouraged by different mathematical standards for teaching [24, 31–38] and recommendations for the preparation of teachers of mathematics [23, 39, 40]. As will be shown below through the discussion of Problems 1 and 2, each of the three methods used in problem solving – symbolic, visual, experimental – may have different forms thereby allowing learners of mathematics to become familiar with multiple techniques within a single method. A teacher who believes that teaching multiple methods of solution of a single problem leads to confusion of students, especially those (sometimes mistakenly) identified as lower-ability learners of mathematics [41], is similar to what a sociologist refers to as a participant observer who may not be aware of rival factors, both external and internal [28].

Likewise, in mathematical problem solving, those factors may be external to a problem including the teacher's choice of technology (tactile or digital) to use, awareness of students' natural interest to learn, intuitive understanding "that it is precisely the student who is most distracted in his class who may, in fact, be his most attentive student" [42] (p. 125), ability "to make mathematical connections between various approaches to solving problems" [39] (p. 31), knowledge of national/local standards and recommendations for teaching mathematics, appreciation of the diversity of problem-solving techniques. Factors internal to a problem may include teacher's choice of mathematical machinery, the use of conceptual shortcuts [43], readiness "to be on the lookout for incomplete or invalid arguments" [23] (p. 33), skills in geometrization of arithmetic and algebraization of geometry (*e.g.*, according to [24] (p. 75)), help students "use coordinates to prove simple geometric theorems algebraically"), ability to recognize in the extension of the problem "the craft of task design" [23] (p. 65), and the use of online sources of information. Some of these external and internal factors are mentioned in the following comment by an elementary teacher candidate, "*I believe in current teaching and learning mathematics standards these types of problems are encouraged in a sense. It seems educators are pushing beyond the boundaries of confinement of sticking to the "one answer approach." With the use of technology, students have much more information that is accessible and are constantly learning and evolving. With this comes a greater diversity of problem-solving."* At the same time, another teacher candidate considers problems with more than one correct answer "*to be tricks, and that is not our goal. Students are under enough stress, they should not have to worry if they found enough correct answers*." Indeed, students should not be in search for a number of correct answers prescribed by a teacher; they just need to be aware of the diversity of strategies that yield such answers. Yet, mathematics all consists of tricks and a student, first seeing the teacher's demonstrations as tricks (*e.g.*, see Problem 2 below), after clarification and explanation by the latter, integrates them into an individual "bag of tricks" [23] (p. 59) to be used in problem solving as appropriate.

3.1 A simple problem with a room for triangulation

As the first illustration of how external and internal factors can structure the process of triangulation in mathematical problem solving, consider

Problem 1

Perimeter of triangle with three consecutive integers serving as its side lengths is equal to 78 linear units. Find the side lengths.

Discussion

Numerically, the problem seeks three consecutive natural numbers with a given sum. Its algebraic solution requires one to possess some BCU, namely, that any three consecutive natural numbers form an arithmetic progression with difference one. Thus, the three numbers can be written as n, $n + 1$, and $n + 2$ from where the equation $n + (n + 1) + (n + 2) = 78$, then the value $n = 25$, and, finally, the triple of integers (25, 26, 27) follow. This method of solving the problem is purely symbolic; it does not discuss what makes the problem solvable, that is, what makes the symbol(s) work. In geometric sense, there is no discussion whether such triangle exists; that is, whether the triple of numbers found satisfy the triangle inequality – any side length is smaller than the sum of the other two side length. The triangle does exist as the largest side length $27 < 25 + 26$. However, changing 78 by 77 or 79 makes the problem numerically unsolvable within the given conditions. This raises another question: Can one modify the problem within the same conceptual structure – having the sum of three or more (extending triangle to a polygon with k sides, $k > 3$) integers in arithmetic progression – in order to make 77 or 79 work? It is through the idea of triangulation in mathematics education as using more than one way of solving a problem and more than one tool in support of problem solving that this (seemingly 'simple') question would be answered below (Remark 1) by the recourse to ACU, thus leading to a new insight through an intelligent reflection on the pretty routine problem-solving method. As Denzin [28], in the context of social science research, put it, "Problems and questions, not theory, create new perspectives" (p. 55). The same is true for mathematics, in general, and its educational field, in particular. In fact, questions are the major means of learning mathematics and, according to [31] (p. 109), "Students' natural inclination to ask questions must be nurtured ... [even] when the answers are not immediately obvious". In the words of a teacher candidate, "*It is okay not knowing the answer to the question but it is not okay with leaving that question* *unanswered ... [thereby not allowing students] to participate in some profound thinking"*. It is through such thinking, when students' questions are addressed by teachers, new problems become posed and solved. As a result, iterative duality of problem solving and posing serves as an agency of triangulation in mathematics education.

3.2 Visual strategy as geometrization of arithmetic

Following the recommendation by the Association of Mathematics Teacher Educators concerned with "unpacking multiple approaches to common mathematical tasks" [39] (p. 90) to which Problem 1 belongs, note that it can be solved differently, in a purely visual way, through what may be called geometrization of arithmetic. In order to carry out this method of triangulation, one needs to possess ACU of the problem structure. Such understanding can be developed through providing explanation as to why the sum of three consecutive terms of any arithmetic sequence (including natural number sequence) is a multiple of three. Consider the diagram of Figure 1 which, by using the "first order symbols ... directly denoting objects or actions" [44] (p. 115), demonstrates that any 3-step staircase representing the sums of three consecutive natural numbers and, in general, integers in arithmetic progression can be rearranged into a three-layer rectangular podium regardless of the length of the upper step. That is, one can perceive abstract symbol n used in the algebraic solution as a concrete (particular) concept embedded into the context of straightening out a three-step staircase of a special type. This, of course, requires the ability to contextualize by probing into the referents provided by the first order symbols. Such ability is also an important aspect of computational thinking – "choosing an appropriate representation for a problem or modeling the relevant aspects of the problem to make it tractable" [20] (p. 33). That is, the ideas behind triangulation and computational thinking in mathematical problem solving go hand-in-hand.

Figure 1 Visual validity: the sum of three consecutive integers is divisible by three.

3.3 Using *Wolfram Alpha* as an online source of information

Another method of solving Problem 1 is to use *Wolfram Alpha* by solving a three-variable equation

$$a + b + c = 78 \tag{1}$$

to see that among a multitude of answers (Figure 2 includes the "More solutions" button) the program provides the triple (25, 26, 27). This problem-solving method contributes to the idea of triangulation in a very straightforward way by validating the use of other methods and demonstrating the computational power of *Wolfram Alpha* as a symbolic calculator. However, even this method requires BCU of variables involved that can be demonstrated through the use of inequalities among the variables in order to minimize the number of triples that the program provides. This is a critical step in using technology as a problem-solving tool because understanding that additional conditions may be used to simplify the search for an answer by the tool might not be obvious for teacher candidates. A collaterally creative [45] question that one may ask is: Why does *Wolfram Alpha* begin with the triple (27, 26, 25)?

To clarify, note that without the use of inequalities

$$a > b > c > 0 \tag{2}$$

among the variables, the first triple to be generated is (1, 1, 76); that is, the program begins with the smallest value of the three numbers satisfying equation (1) which, however, does not guarantee that the triangle inequality holds. Indeed, $76 > 1 + 1$. Nonetheless, in the presence of inequalities (2), the smallest value of a satisfying equation (1) is 27. Perhaps the program begins by dividing 78 by 3 yielding the triple (26, 26, 26) but then increases (decreases) the first (third) term by 1. This understanding of a possible systematic reasoning behind a computational algorithm is not just critical for the development of computational thinking by using heuristic, recursive reasoning, error correction, and, ultimately, triangulation – "a way humans solve problems" [20] (p. 35), but it enables one to solve similar problems without the use of technology. Moreover, communicating to *Wolfram Alpha* that the three variables differ by one, that is, adding the condition "$a – b = 1$, $b – c = 1$" to (2) yields exactly the triple (27, 26, 25) sought by Problem 1.

Figure 2 (in which the triangle inequality $a < b + c$ augments the command entered into the input box of *Wolfram Alpha*) shows how, starting from 27 as the largest element of the triple of consecutive integers with the sum 78, *Wolfram Alpha*, by increasing 27 by one, step by step, provides two triples of not consecutive integers with the sum 78, then four triples with this sum, and so on ... until the last triple (38, 37, 3), satisfying the triangle inequality, is reached. Figure 3 shows how such alteration of triples by increasing the largest term works with concrete materials, in the general case of n, $n – 1$, $n – 2$ when increasing n blocks by one block yields two alterations of the other two blocks of the staircase.

The use of multiple methods enables one to better understand how a particular method works. In mathematics education, the concept of triangulation provides an opportunity to understand how a computer program deals with symbolic computation. Such analysis of the use of a digital tool contributes to the development of residual mental power that can be used in the absence of the tool [46]. In general, the goal of such pedagogy is to ensure that today's collaboration with a 'more knowledgeable other' [18] would facilitate an independent performance tomorrow.

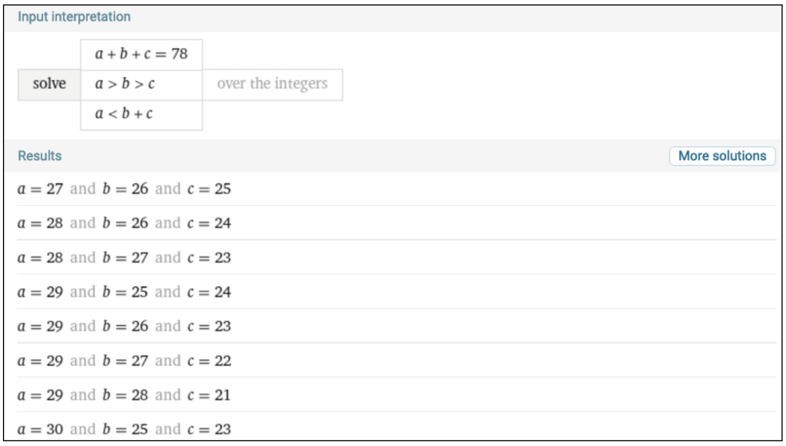

Figure 2 Using *Wolfram Alpha* to get the triple (25, 26, 27)

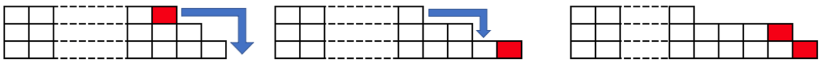

Figure 3 Visual demonstration: from (n, $n – 1$, $n – 2$) to ($n + 1$, $n– 1$, $n – 3$) to ($n + 1$, n, $n – 4$).

3.4 From trial-and-error method to posing the problem's extension

Whereas the strategies discussed above made it clear that, as mathematics educators in England, reflecting on their teaching experience, advised "there was more than one way of doing things" [47] (p. 18), Problem 1 can yet be solved in the context of a spreadsheet (Figure 4) through computational trial and error. To this end, one can attach a slider to cell A3, define two consecutive (to A3) integers in cells A4 and A5 and defining the formula =sum(A3:A5) in cell A1, to alter the entry A3 until cell A1 displays 78. One can recognize that the sums displayed in cell A1 represent the sequence 6, 9, 12, 15, 18, ... This recognition, motivated by numerical evidence of the spreadsheet-based computational experiment not available within other methods (except using *Wolfram Alpha*), can prompt one to extend Problem 1 to the case when three (or more) consecutive integers are terms of an arithmetic sequence with the first term m. The sum of k terms of such a sequence, representing perimeter of a polygon with k sides, is equal to $\frac{(2m+k-1)k}{2}$. When the sum is equal to 78, we have the equation $\frac{(2m+k-1)k}{2} = 78$ whence $(2m + k - 1)\,k = 156$. That is, a new method is based on the factorization of the double of the given sum in two integer factors. Possible factorizations of 156 (assuming $k > 2$; indeed, representing a natural number as a sum of two like numbers is either trivial or impossible, let alone no polygon with two sides exists) are $156 = 3 \times 52 = 4 \times 39 = 6 \times 26 = 12 \times 13$ where the first factor points at the number of terms. Therefore, the case $k = 3$ yields $2m + 3 – 1 = 52$ whence $m = 25$ and the triple is (25, 26, 27). The case $k = 4$ (quadrilateral) yields $2m + 4 – 1 = 39$ whence $m = 18$ and the quadruple is (18, 19, 20, 21). The case $k = 6$ (hexagon) yields $2m + 6 – 1 = 26$ whence $2m = 21$ indicating that something is wrong with this case and, more generally, the very method has a drawback. Nonetheless, the case $k = 12$ (dodecagon) yields $2m + 12 – 1 = 13$ whence $m = 1$ and the duodecuple is (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12). How can one explain the case $k = 6$?

We see that some methods used in triangulation may not work well and this raises a new question about the method itself. That is, we may talk about the need for what in McFee (1992) [11] was called triangulation *within* a method, something that in the context of mathematics education may include somewhat unobtrusive properties and, therefore, the factorization method requires a deeper analysis into the problem-solving technique applied to Problem 1. For example, if the number 78 is replaced by the number 81 we would not have come across a case that does not work. Indeed, factoring 162 in two integer factors (greater than two) yields $162 = 3 \times 54 = 6 \times 27 = 9 \times 18$ from where the following three solutions – triangle, hexagon, and nonagon – result: (26, 27, 28), (11, 12, 13, 14, 15, 16) and (5, 6, 7, 8, 9, 10, 11, 12, 13).

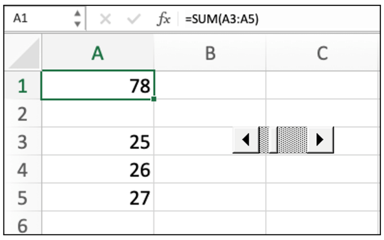

Figure 4 Using a spreadsheet to get the answer through trial and error

3.4.1 Remark 1

If we extend the discussion to have the sum of *k* integers in arithmetic progression with the first term *m* and difference $d \geq 1$, then such sum is equal to $\frac{(2m+d(k-1))k}{2}$. Exploring this expression in the context of the above-mentioned modification of Problem 1 to have positive integers in arithmetic progression with the sum 81 would yield 31 solutions (26 triples, three 6-tuples, and two 9-tuples; consecutive integers representatives from each of the three cases were listed above), with the sum 77 (mentioned at the beginning of the discussion) three 7-tuples ($77 = 8 + 9 + 10 + \cdots + 14 = 5 + 7 + 9 + \cdots + 17 = 2 + 5 + 8 + \cdots + 20$) and no solutions for a prime number 79 (also mentioned at the beginning of the discussion; its double can only be factored with itself as a factor). The possibility of extending a problem to include new ideas and features is an important facet of using triangulation in mathematical problem solving and posing. It is through triangulation, that not only the validation of solution can be achieved but the deficiency of some problem-solving methods and ways of using them can be recognized and refined. This issue will be discussed in Section 4.3.

3.4.2 Remark 2

With regard to the above case $k = 6$ that yielded no solution, one can say that applying the concept of triangulation aimed at validating the correctness of a certain claim may result in realization that some methods include hidden pitfalls which only can be revealed through the application of a particular method to different data. In the context of the factorization method, the equality $(2m + k - 1)\,k = 2 \times N$ works only when the factors in the left-hand side are of different parity. Indeed, the factors $2m + k - 1$ and k may not be of the same parity because $2m$ is always even and if k is odd, then $k - 1$ is even thus is $2m + k - 1$ even; if k is even, then $k - 1$ is odd thus $2m + k - 1$ is odd. The issue of computational thinking to be discussed is how to improve a particular triangulation method to make it work in all situations. For example, one can narrow down the search using *Wolfram Alpha* in order to find pairs of factors of different parity, not just any two factors forming a given integer. To this end, one can enter into the input box of the program the command "solve over the integers (2m+1)(2n) = 78". Figure 5 shows how this command results in three factorizations only: $156 = 3 \times 52 = 4 \times 39 = 12 \times 13$. But this was a result of a deeper insight into the factorization method. Furthermore, in the case $d > 1$, the equality $(2m + d(k-1))\,k = 2 \times N$ works when the factors in its left-hand side are of the same parity. Yet, the inequality between the factors remains the same, *i.e.*, $2m + d\,(k-1) > k,\ m \geq 1,\ d \geq 1,\ k \geq 3$. Indeed, $2m + d\,(k-1) - k = 2m + (d-1)\,(k-1) - 1 \geq 2m - 1 > 0$.

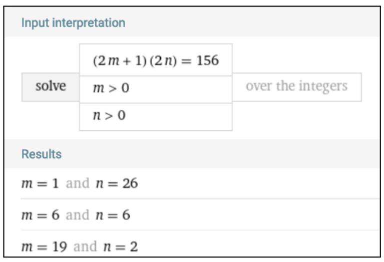

Figure 5 Factorization ensuring different parity of factors

4 Triangulation in trigonometry

As mentioned by one of the secondary teacher candidates, "*Part of the beauty about mathematics is that problem solving can be accomplished in more than one way. I have prior experience* *with tutoring and student teaching when I liked to show my students different ways of solving problems to have them gain a higher level of appreciation of mathematics.*" This section provides another illustration of triangulation in mathematical problem solving intended to demonstrate what the candidate might have had in mind. The candidate's focus on developing secondary students' appreciation of mathematics is commendable, especially when not just subject matter of mathematics, but education, in general, is "so frequently disliked" [48] (p. 2). Whereas the illustration deals with a trigonometric equation the first solution of which (Section 4.1) is pretty straightforward and does not require any sophisticated use of trigonometric identities as it is often the case with trigonometric equations, the equation to be considered allows one not only to apply different methods of problem solving, but, in addition, to triangulate between the methods because each method provides a symbolically distinctive answer. This outcome requires what may be called the second order triangulation aimed at deciding the validity of triangulation of the first order defined as the application of different solution methods to the same problem. To begin, consider

Problem 2

Solve the equation

$$2\sin x + 3\cos x = 1 \tag{3}$$

on the segment $[-\pi,\ \pi]$.

Discussion

In what follows, three different methods of solving equation (3) will be considered. Unlike Problem 1, this time a symbolic form of answer in each case would be different. One of the methods (Section 4.3) would yield an extraneous answer and in order for the latter to be recognized, one has to possess ACU. The reason for this difference between Problem 1 and Problem 2 is in the uniqueness of trigonometry that deals with angles which have multiple forms of representation through arc functions. Already the right angle can be represented as $90°$, $\frac{\pi}{2}$, $sin^{-1}1$, $cos^{-1}0$, $1.570796\ldots$ (non-terminating non-repeating decimal), and even as $[1; 1,\ 1,\ 3,\ 31,\ 1,\ \ 145,\ \ldots]$ (continued fraction). This is similar to how the fraction 1/2 can be represented as 0.5 and $0.4\bar{9}$ (decimals), 50% (percent), and [0; 2] (continued fraction).

4.1 An auxiliary angle method

The first problem-solving technique deals with writing the left-hand side of equation (3) in an equivalent form $\frac{2}{\sqrt{13}}\sin x + \frac{3}{\sqrt{13}}\cos x = \frac{1}{\sqrt{13}}$ (towards replacing a linear combination of two trigonometric functions by a single function) which, after noting that the equality $13 = 2^2 + 3^2$ implies $(\frac{2}{\sqrt{13}})^2 + (\frac{2}{\sqrt{13}})^2 = 1$, allows for the following introduction of an auxiliary angle α. Setting, $\frac{2}{\sqrt{13}} = \cos\alpha$, $\frac{3}{\sqrt{13}} = \sin\alpha$ whence $\tan\alpha = 3/2$ and $\alpha = \tan^{-1}\frac{3}{2}$, the last modification of equation (3) can be rewritten as $\cos\alpha \times \sin x + \sin\alpha \times \cos x = \frac{1}{\sqrt{13}}$ or $\sin(x+\alpha) = \frac{1}{\sqrt{13}}$ whence equation (3), on the segment $[-\pi,\ \pi]$ has two roots

$$x = -tan^{-1}\frac{3}{2} + sin^{-1}\frac{1}{\sqrt{13}} \tag{4}$$

and

$$x = \pi - (tan^{-1}\frac{3}{2} + sin^{-1}\frac{1}{\sqrt{13}}) \tag{5}$$

A graphical solution of equation (3), supported by the *Graphing Calculator* [49], is shown in Figure 6. One can see that the graph of the function $y = 2\sin x + 3\cos x - 1$ has two points of intersection with the *x*-axis on the segment $[-\pi,\ \pi]$. The points reside at different sides of the *y*-axis indicating that formulas (4) and (5) define, respectively, (the smallest in absolute value) negative and positive roots of equation (3). In particular, because (by definition, considered as BCU of trigonometry and, perhaps, as ACU, in general) both $tan^{-1}\frac{3}{2}$ and $sin^{-1}\frac{1}{\sqrt{13}}$ reside inside the segment $\left[0,\ \frac{\pi}{2}\right]$, in addition to graphical evidence provided by Figure 6, the inequalities

$$0 < sin^{-1}\frac{1}{\sqrt{13}} + tan^{-1}\frac{3}{2} < \pi \tag{6}$$

hold true. As it is always the case when inequalities are used instead of equalities, the smaller the interval within which something (to be estimated) is included, the better the estimate (a substitute for the exact value) of this something. In particular, π as the upper estimate of the sum in inequalities (6) can be replaced by $\frac{\pi}{2}$. To this end, note that, as shown in Figure 7,

$$sin^{-1}\frac{1}{\sqrt{13}} + tan^{-1}\frac{3}{2} < sin^{-1}\frac{2}{\sqrt{13}} + tan^{-1}\frac{3}{2} = \frac{\pi}{2}$$

and therefore, inequalities (6) can be refined (in fact, significantly improved) as follows:

$$0 < sin^{-1}\frac{1}{\sqrt{13}} + tan^{-1}\frac{3}{2} < \frac{\pi}{2} \tag{7}$$

In particular, inequalities (7) when applied to formula (5) suggest that the latter defines an angle residing inside the segment $[\frac{\pi}{2},\ \pi]$. Put another way, without inequalities (7) and the graphical solution of equation (3), one can only conclude that formula (5) defines a positive root of the equation. This was the reason to mention that inequalities (7) significantly improved inequalities (6).

Once the value of a new result (finding) is recognized, its alternative validation can be seen, once again, as a triangulation *within* a method [11]. That is, assuming that proving is a method, using more than one proof is a triangulation aimed at the validity of the method. Inequalities (7) can be proved differently. Indeed, $sin^{-1}\frac{1}{\sqrt{13}} < sin^{-1}\frac{1}{2} = \frac{\pi}{6}$ and $tan^{-1}\frac{3}{2} < tan^{-1}\sqrt{3} = \frac{\pi}{3}$ (because, in the first quadrant, the larger the value of sine (or tangent), the larger the angle). Therefore, the sum of arc functions in (6) can be estimated as follows:

$$\sin^{-1}\frac{1}{\sqrt{13}} + \tan^{-1}\frac{3}{2} < \sin^{-1}\frac{1}{2} + \tan^{-1}\sqrt{3} = \frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{2}$$

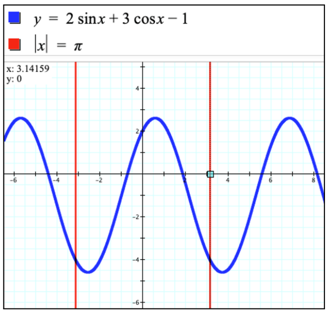

Figure 6 Graphical solution of equation (3) within the range $|x| < \pi$

4.1.1 Remark 3

As mentioned by Denzin [28] in the context of triangulated perspective on the methods of sociology, because "many propositions combine concepts that have no empirical referents ... propositions should be combined with other propositions so that a deductive theoretical system may be developed" (pp. 44, 45). This perspective is true for mathematical education as well. Indeed, whereas the sum in (7) has no empirical referent, it is through the use of inequalities as tools of deductive reasoning in mathematics, that Figure 7 serves as an empirical referent for this sum.

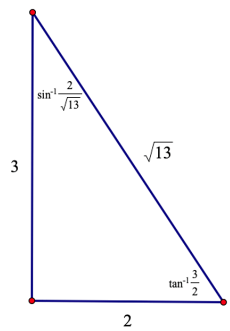

Figure 7 Demonstrating that $\sin^{-1}\frac{2}{\sqrt{13}} + \tan^{-1}\frac{3}{2} = \frac{\pi}{2}$

4.2 A method of reduction to a homogeneous equation

The second method of solving equation (3) is to re-write it in terms of $\sin\frac{x}{2}$ and $\cos\frac{x}{2}$ as the homogeneous equation

$$2 \times 2\sin\frac{x}{2}\cos\frac{x}{2} + 3\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right) = \cos^2\frac{x}{2} + \sin^2\frac{x}{2}$$

and then divide both sides by $\cos^2\frac{x}{2}$, ensuring the preservation of equivalence as $\cos\frac{x}{2} \neq 0$ (otherwise, $\sin\frac{x}{2} = 1$ and the last equation would have unequal left- and right-hand sides, -3 and

1, respectively), to get $4\tan\frac{x}{2}+3\left(1-\tan^2\frac{x}{2}\right)=1+\tan^2\frac{x}{2}$ or $2\tan^2\frac{x}{2}-2\tan\frac{x}{2}-1=0$, or $\tan\frac{x}{2}=\frac{1\pm\sqrt{3}}{2}$, whence

$$x=2\tan^{-1}\left(\frac{1+\sqrt{3}}{2}\right) \tag{8}$$

or

$$x=2\tan^{-1}\left(\frac{1-\sqrt{3}}{2}\right) \tag{9}$$

4.2.1 Remark 4

Note that trigonometry as a branch of mathematics provides many fine features demonstrating the need for rigor in advancing a particular reasoning technique. Mathematical rigor, in general, demonstrates how validity of statements in the particular context of trigonometry (using the language of social science research) "depend[s] on how thoroughly and defensibly or correctly this [reasoning] has been done" [30] (p. 346). Indeed, in many cases, trigonometry included, dividing both sides of an equation (or inequality) by a variable expression does not preserve the equivalence of relations and leads to unwanted loss or/and gain of solutions (*e.g.*, Abramovich & Ehrlich (2007) [50]). It is due to rigor of preserving equivalence when solving an equation that formulas (8) and (9) define the roots and only the roots of equation (3).

The next step in the process of triangulation is to demonstrate that results obtained through different problem-solving techniques are numerically identical despite being symbolically different. That is, triangulation of the second order has to be used. To this end, the following two identities have to be proved:

$$2tan^{-1}(\frac{1+\sqrt{3}}{2})=\pi-(tan^{-1}\frac{3}{2}+sin^{-1}\frac{1}{\sqrt{13}}) \tag{10}$$

and

$$2tan^{-1}(\frac{1-\sqrt{3}}{2})=-tan^{-1}\frac{3}{2}+sin^{-1}\frac{1}{\sqrt{13}} \tag{11}$$

Put another way, proving identities (10) and (11) may be seen as problems posed in the context of triangulation of the second order.

When the author studied trigonometry in high school, technology was not available and proving identities (10) and (11) was purely formal supported by trigonometric identities, trigonometry of right triangles, and definitions of the principal values of arc functions (all requiring ACU of school mathematics). In the digital era, proof of identities (10) and (11) can be outsourced to *Wolfram Alpha* or *Maple* (Figures 8, 9 and 10). As mentioned in Langtangen & Tveito (2001) [51] (pp. 811-812), "Much of the current focus on algebraically challenging, lengthy, error-prone paper and pencil work can be significantly reduced. The reason for such an evolution is that the computer is simply much better than humans on any theoretically phrased well-defined repetitive operation". Nonetheless, one has to triangulate between at least two digital tools in order to validate a computational result. This position is similar to the concept of data triangulation which, in addition to including more than one individual in research, indicates that time has to be included [28]. For example, just as the use of programming codes may change with the appearance of a new version of software and thus published information about codes used in computations is accurate to the extent of the time a paper was written, within the specific context of research on continuing education "to study the effect of an inservice program on teachers, one should observe teachers at different times of the school day or year" [29] (p. 14). Figure 10 shows the use of *Maple* in verifying (at the *time* of writing this paper) confirmation of identities (10) and (11) by *Wolfram Alpha*.

4.2.2 Remark 5

Not only proving numeric identities (10) and (11) can be outsourced to *Wolfram Alpha* (or *Maple*), but further refinement of numeric inequalities (7) can be done by the tool as well by using trial and error. Figure 11 shows that the coefficients in π can be increased and decreased, respectively from 0 and 1/2 to 0.402 and 0.403 thus locating on the number line the position of the sum $\sin^{-1}\frac{1}{\sqrt{13}}+\tan^{-1}\frac{3}{2}$ to the accuracy of $0.001\pi\cong 0.003$. This raises the question: Why do we need to be skillful in using inequalities when estimations (or even computations) can be done by a computer? An answer to this question was already given in the introduction – just solving a simple mathematical problem using technology sometimes requires competence beyond BCU. Another such example is shown in Figure 12. The *Graphing Calculator*, when asked to solve the equations $2\sin x+3\cos x=5$ and $\sqrt{x+20}=x$, provides the same response, "Not satisfied in the region shown", yet for very different reasons. Whereas the former equation does not have real solutions as the largest value of its left-hand side is $\sqrt{13}<5$, the (real) root, $x = 5$, of the latter equation is simply not shown in the region selected. That is, without understanding of mathematical situation involved and its connection to computational tools used, the use of technology in problem solving may lead astray. Depending on a situation, either BCU or ACU is required to connect mathematical and computational thinking [20] in order to avoid misconceptions.

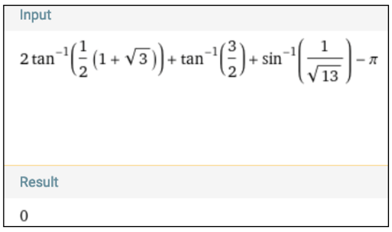

Figure 8 *Wolfram Alpha*'s verification of relation (10)

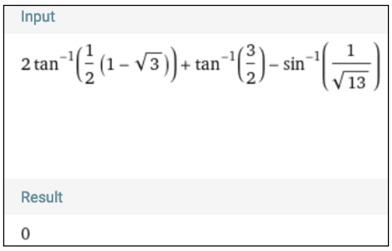

Figure 9 *Wolfram Alpha*'s verification of relation (11)

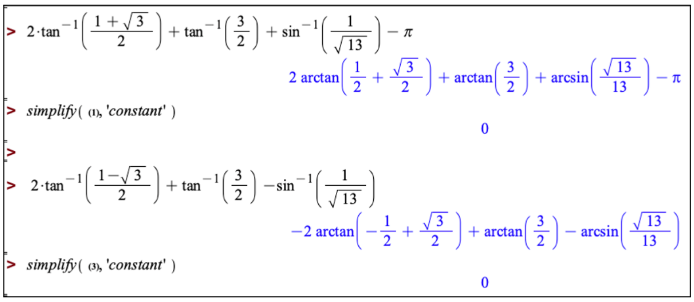

Figure 10 Using *Maple* to validate computations by *Wolfram Alpha*

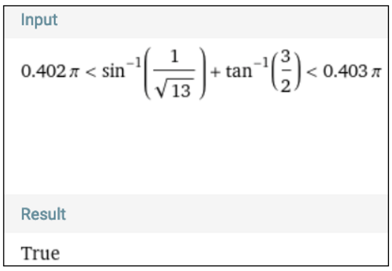

Figure 11 *Wolfram Alpha*'s refinement of inequalities (7)

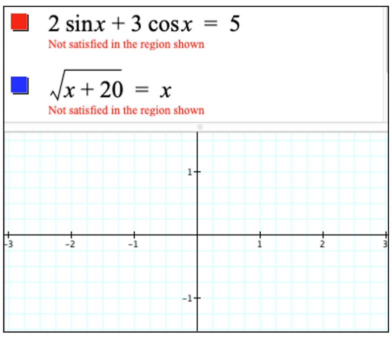

Figure 12 Declining solution for different reasons

4.3 A method of reduction to algebraic equations

Once again, we begin this section by citing another secondary teacher candidate who believes that "*showing multiple solutions helps students realize mathematics is more dynamic and by tackling more complex mathematics at the added level of abstraction students mature faster*." In other words, mathematical triangulation as a pedagogical approach can potentially make mathematics education less static in the perception of problem solvers. The third method of solving equation (3) may serve as an example of unforeseen dynamism of mathematical complexity the abstractness of which can be eased through the use of multiple digital tools. To illustrate, one can set $\sin x = a,\ \cos x = b$, and consider the system of equations and inequalities

$$2a + 3b = 1, a^2 + b^2 = 1, |a| \leq 1, |b| \leq 1 \tag{12}$$

from where a and b have to be found. Substituting $b = \frac{1-2a}{3}$ in the second equation of (12) yields $a^2 + \left(\frac{1-2a}{3}\right)^2 = 1$ or $13a^2 - 4a - 8 = 0$ whence $a = \frac{2\pm 6\sqrt{3}}{13}, b = \frac{3\mp 4\sqrt{3}}{13}$. That is, $a^2 + \left(\frac{1-2a}{3}\right)^2 = 1$ or $13a^2 - 4a - 8 = 0$ whence $a = \frac{2\pm 6\sqrt{3}}{13}, b = \frac{3\mp 4\sqrt{3}}{13}$. In terms of arc functions, we have

$$x = \sin^{-1}\left(\frac{2+6\sqrt{3}}{13}\right) \text{ and } x = \cos^{-1}\left(\frac{3-4\sqrt{3}}{13}\right) \tag{13}$$

or

$$x = \sin^{-1}\left(\frac{2-6\sqrt{3}}{13}\right) \text{ and } x = \cos^{-1}\left(\frac{3+4\sqrt{3}}{13}\right) \tag{14}$$

Simply by counting the number of roots, one can see that (13) and (14) include twice as many roots as (4) and (5), as well as (8) and (9). This means that, apparently, formulas (13) and (14) include extraneous roots. Consequently, the questions to be answered are: Where do extraneous roots come from? Why did two previous methods of solving equation (3) yield no extraneous roots?

The graphs of Figure 13 show that between two values of x listed in (13) only one value coincides with the positive root of equation (3). In order to find out which one is extraneous (or, at least, which one includes an extraneous value), note that $\sin x = \frac{2+6\sqrt{3}}{13}$ twice on $[0,\ \pi]$: when $x < \frac{\pi}{2}$ and when $x > \frac{\pi}{2}$; yet only in the latter case $\cos x < 0$. That is, only the second value of x in (13) is the right answer. Likewise, the graphs of Figure 14 show that between two values of x listed in (14), only one value coincides with the negative root of equation (3) on the segment $[-\pi,\ \pi]$. In order to find out which one is extraneous (or, at least, includes an extraneous value), note that $\cos x = \frac{3+4\sqrt{3}}{13}$ twice on $[-\frac{\pi}{2},\ \frac{\pi}{2}]$: when $x > 0$ and when $x < 0$; yet only in the latter case $\sin x < 0$. That is, only the first value of x in (14) is the right answer. With regard to formulas (8) and (9), one can check to see (Figure 15) that

$$\cos^{-1}\left(\frac{3-4\sqrt{3}}{13}\right) = 2\tan^{-1}\left(\frac{1+\sqrt{3}}{2}\right) \tag{15}$$

and

$$\sin^{-1}\left(\frac{2-6\sqrt{3}}{13}\right) = 2\tan^{-1}\left(\frac{1-\sqrt{3}}{2}\right). \tag{16}$$

With regard to formulas (4) and (5), one can check to see (Figure 16) that formula (4) is identical to either side of identity (16) and formula (5) is identical to either side of identity (15).

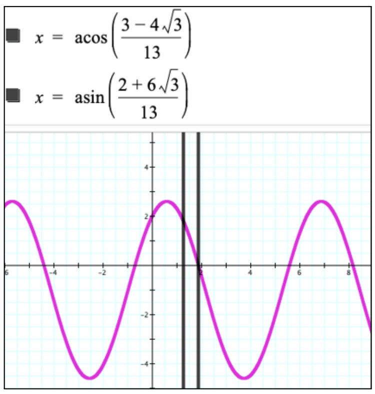

Figure 13 Graphs of (13) along with the graph of equation (3)

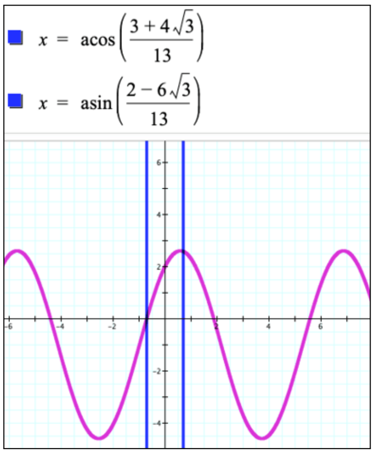

Figure 14 Graphs of (14) along with the graph of equation (3)

$$> \cos^{-1}\left(\frac{3-4\cdot\sqrt{3}}{13}\right) - 2\cdot\tan^{-1}\left(\frac{1+\sqrt{3}}{2}\right)$$
$$\pi - \arccos\left(-\frac{3}{13} + \frac{4\sqrt{3}}{13}\right) - 2\arctan\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right)$$
$$> \mathit{simplify}(\ (1), \text{'constant'}\ )$$
$$0$$
$$> \sin^{-1}\left(\frac{2-6\cdot\sqrt{3}}{13}\right) - 2\cdot\tan^{-1}\left(\frac{1-\sqrt{3}}{2}\right)$$
$$-\arcsin\left(-\frac{2}{13} + \frac{6\sqrt{3}}{13}\right) + 2\arctan\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\right)$$
$$> \mathit{simplify}(\ (3), \text{'constant'}\ )$$
$$0$$

Figure 15 Verifying numeric identities (15) and (16) using *Maple*

$$> -\sin^{-1}\left(\frac{2-6\cdot\sqrt{3}}{13}\right) - \tan^{-1}\left(\frac{3}{2}\right) + \sin^{-1}\left(\frac{1}{\sqrt{13}}\right)$$
$$\arcsin\left(-\frac{2}{13} + \frac{6\sqrt{3}}{13}\right) - \arctan\left(\frac{3}{2}\right) + \arcsin\left(\frac{\sqrt{13}}{13}\right)$$
$$> \mathit{simplify}(\ (1), \text{'constant'}\ )$$
$$0$$
$$> \cos^{-1}\left(\frac{3-4\cdot\sqrt{3}}{13}\right) + \tan^{-1}\left(\frac{3}{2}\right) + \sin^{-1}\left(\frac{1}{\sqrt{13}}\right) - \pi$$
$$-\arccos\left(-\frac{3}{13} + \frac{4\sqrt{3}}{13}\right) + \arctan\left(\frac{3}{2}\right) + \arcsin\left(\frac{\sqrt{13}}{13}\right)$$
$$> \mathit{simplify}(\ (3), \text{'constant'}\ )$$
$$0$$

Figure 16 Comparing (15) and (16) to (5) and (4), respectively, using *Maple*

4.3.1 Remark 6

Because, by definition, $\sin^{-1} x \in \left[0\,,\ \frac{\pi}{2}\right]$ for $0 \le x \le 1$, where $\cos x \ge 0$, the second root defined by formulas (13), may be expressed as the difference $\pi - \sin^{-1}\left(\frac{2+6\sqrt{3}}{13}\right) \in \left(\frac{\pi}{2}, \pi\right)$. In particular, using *Maple* or *Wolfram Alpha* (or both), one can check that $\pi - \sin^{-1}\left(\frac{2+6\sqrt{3}}{13}\right) = \cos^{-1}\left(\frac{3-4\sqrt{3}}{13}\right)$ or $\sin^{-1}\left(\frac{2+6\sqrt{3}}{13}\right) + \cos^{-1}\left(\frac{3-4\sqrt{3}}{13}\right) = \pi$. Likewise, because, by definition, $\cos^{-1} x \in \left[0, \frac{\pi}{2}\right]$ for $0 \le x \le 1$, where $\sin x \ge 0$, the first root defined by formulas (14), $\sin^{-1}\left(\frac{2-6\sqrt{3}}{13}\right) \in \left(-\frac{\pi}{2}, 0\right)$, may be expressed as $-\cos^{-1}\left(\frac{3+4\sqrt{3}}{13}\right) \in \left(-\frac{\pi}{2}, 0\right)$. In particular, using *Maple* or *Wolfram Alpha* (or both), one can check that $\sin^{-1}\left(\frac{2-6\sqrt{3}}{13}\right) = -\cos^{-1}\left(\frac{3+4\sqrt{3}}{13}\right)$ or $\cos^{-1}\left(\frac{3+4\sqrt{3}}{13}\right) + \sin^{-1}\left(\frac{2-6\sqrt{3}}{13}\right) = 0$. In addition, a useful technique in carrying out triangulation is to prove in the traditional (paper-and-pencil) fashion that in a right triangle with the side lengths of the hypothenuse and a leg equal to, respectively, 13 and $6\sqrt{3} - 2$, the side length of the second leg (as shown in Figure 17) is

$$\sqrt{13^2 - (6\sqrt{3} - 2)^2} = \sqrt{57 + 24\sqrt{3}} = \sqrt{9 + 48 + 2 \times 3 \times 4\sqrt{3}}$$
$$= \sqrt{3^2 + (4\sqrt{3})^2 + 2 \times 3 \times 4\sqrt{3}} = \sqrt{(3 + 4\sqrt{3})^2} = 3 + 4\sqrt{3}$$

Although both *Maple* and *Wolfram Alpha* in response to $\sqrt{57+24\sqrt{3}}$ yield $3+4\sqrt{3}$, the context of triangulation provides an opportunity for the traditional recognition of full square in an irrational radicand, thereby "bringing some coherence to the bag of tricks for factoring and completing the square that are traditional in high school algebra" [23] (pp. 59, 60).

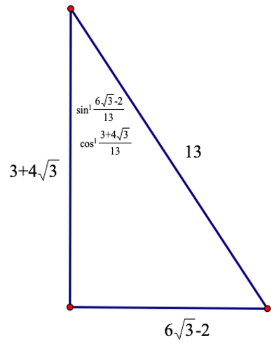

Figure 17 Triangulating the identity $\cos^{-1}\left(\frac{3+4\sqrt{3}}{13}\right)=\sin^{-1}\left(\frac{6\sqrt{3}-2}{13}\right)$

5 Conclusion

This paper was written to introduce the notion of triangulation used by social science researchers since the mid of the last century [26, 27] into the context of mathematics education as a field of disciplined inquiry [52, 53]. The paper suggested that triangulation in mathematics education can be interpreted as the use of multiple ways of solving a problem. In the digital era, triangulation in mathematics education includes using more than one software program in confirming the result of a computational experiment. In that way, triangulation supports the development of computational thinking [20] through deciding the appropriate digital accommodations for representation of a problem and its possible extension. Whereas different ways of solving a problem enable learners of mathematics to arrive at the same answer through different solution strategies, it is possible that the variation of strategies (that is, triangulation of the first order) might result in the variation of the symbolic forms of the answer. In that case, triangulation of the second order has to be used. The latter may be as simple as demonstrating the equivalence of fractional and decimal representations or may be more involved by demonstrating the equivalence of answers represented through different arc functions. In some cases, triangulation of the second order may include either uncovering an extraneous answer provided by a method and investigating its deficiency or deciding whether an alternative method had a flaw that led to the loss of correct answers.

The paper made connections between rigor as one of the goals of using triangulation in social science research and rigor as an outcome of triangulation in mathematical problem solving, both in research and educational contexts. External and internal factors affecting implementation and support of triangulation in mathematics education by classroom teachers as the major custodians of multiple solution strategies in problem solving were considered. It was demonstrated how problem posing emerges in the context of technology-enhanced triangulation thus making the iterative duality of solving and posing problems by the learners of mathematics an important agent of successful modern-day mathematical teacher preparation.

Reflections of the author's students, pre-service K-12 teachers, were included and analyzed. Triangulation helps learners of the subject matter to better grasp abstract ideas at different levels of complexity. Two more reflections, indicative of the existence of triangulated approaches to problem solving, are worth sharing. In the words of a secondary teacher candidate, "*When variables are viewed in isolation from the totality, they can seem abstract or irrelevant. It really made sense to me with the graph, equation, and coordinates and how those all work together to make the problems visually imaginable.*" In this comment, one can see how multiple perspectives on variables as abstract entities used to formulate and solve equations, like the one discussed in Problem 2, make the symbols relevant through the construction of graphs. Indeed, the graphs of equations are built of specific points the coordinates of which may be seen through Vygotskian lens as "the first order symbols [which give meaning to] the second order symbolism" [44] (p. 115) of variables, otherwise perceived abstract and possibly irrelevant.

Another student of the author, an elementary teacher candidate, reflecting on helping young learners to grasp the abstraction of arithmetic, noted: "*I am a teaching assistant in an elementary school, and usually work with students that struggle. I like how students in our school learn multiple ways to solve problems: i.e.– number bonds, ten frames, 100's charts... It helps kids* *start to understand the concepts of what things like "addition" and "subtraction" actually mean*." Once again, one can see how teachers in the modern-day elementary classroom use what is, in fact, a triangulated approach to the concepts of arithmetic that stems from multiple educational epistemologies supported by different tactile technologies. This approach allows for justification of foundational concepts of mathematics by demonstrating their historical roots using contemporary tools. It is through the grasp of meaning that tactile activities provide, just as great minds of the past struggled with the meaning of knowledge [54], elementary students through their own, not insignificant, cognitive struggles come to know how to show a teacher that they indeed know and understand mathematical knowledge taught to them with professional confidence necessary for triangulation.

The use of triangulation in mathematics education as more than one way of solving a problem uplifts epistemic development of K-12 students and their future teachers. In the digital era, triangulation contributes to teacher candidates' better grasp of mathematical ideas and enhances familiarity with appropriate use of available computer programs. The diversity of tactile tools, both physical and virtual, provides another dimension to the pedagogy of triangulation. Finally, the ideas of triangulation can be extended to a variety of collegiate mathematics courses beyond technology-supported teacher preparation programs the long-term experience with which motivated the author to write this paper.

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