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RESEARCH ARTICLE

From a teacher education course to upper elementary classrooms and back: Revealing innate abilities of children to do teachers' mathematics

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Abstract: One of the key ideas of the modern-day elementary mathematics teacher education deals with mediating learning by visual thinking to enable transition from seeing and acting on concrete objects to describing the visual and the physical through culturally accepted symbolic representations. This paper shares mathematical activities designed originally for teacher candidates and used with students in upper elementary classrooms at a school in Upstate New York with minority student enrollment 97%. Because successful use of conceptual thinking by young students does have positive impact on their future teachers, connection of work in the school to a master's level elementary mathematics education course taught by the authors is discussed. It is shown how using a spreadsheet and *Wolfram Alpha* allows for the research-like extension of the activities to the secondary level of mathematics education.

Keywords: elementary mathematics, conceptual thinking, problem solving, hands-on activities, digital technology, teacher education

1 Introduction

Common Core State Standards [1], one of the major PK-12 educational documents in the United States at the time of writing this paper, aim to achieve coherence of mathematical content and curricula "not only by stressing conceptual understanding of key ideas, but also by continually returning to organizing principles such as place value or the properties of operations to structure those ideas" (p. 4). One of the key ideas of mathematics education deals with mediating learning by visual thinking to enable transition from seeing and acting on concrete objects to describing the visual and the physical through culturally accepted symbolic representations. To this end, elementary teacher candidates "should know ways to use mathematical drawings, diagrams, manipulative materials, and other tools to illuminate, discuss, and explain mathematical ideas and procedures" [2] (p. 33). The best way to learn such teaching practice for the candidates is "to connect their methods experiences with Pre-K-12 classrooms and children" [3] (p. 36).

This paper is written to demonstrate a possible integration of the above three recommendation expressed recently by the major agents of educational advancement in the United States into the authors' work as mathematics teacher educators. One of the authors' ideas behind such integration was to come to a local school with problem-solving activities used in the graduate Elementary Mathematics Content and Methods course (referred to below as the course) they teach to teacher candidates asynchronously. If such idea demonstrates that young children, not necessarily selected for their high-level mathematical abilities, can comprehend mathematical concepts that the candidates might deem too difficult for their future students to grasp, then the connection between the two groups of learners of mathematics is established and the integration of the above-mentioned educational recommendations may be considered successful.

In what follows, this integration will be illustrated against the backdrop of two problems. One problem was motivational for young children as it involved the arrangement of cookies on plates following the rule governed by the recursion that Fibonacci numbers satisfy. More specifically, the context includes several plates that are lined up and the question is to find all ways to end up with a certain number of cookies on the last plate if the number of cookies on the plates beginning from the third one is the sum of cookies on the previous two plates. The simplest version of this problem that was used by the authors with third grade students as a warm-up activity is to deal with three plates only. Then the problem was extended to four plates, both in the 3rd and 5th grade classrooms.

A more complicated scenario used by the authors with future elementary teachers is to find all ways to put cookies on the first two plates if beginning from the third plate the number of cookies is the sum of that number on the previous two plates and through this rule to end up with n cookies on the m-th plate. Posing problems of this general type can be computerized using a spreadsheet (see Section 6 below), with n and m being slider-controlled parameters, in order to develop problems with a single answer, with more than one answer, and with no answer. For example, there is only one way to put eight cookies on the fifth plate (1-2-3-5-8), two ways to put six cookies on the fourth plate (2-2-4-6 and 4-1-5-6), and no way to put six cookies on the fifth plate. This general problem has its origin in a 1995 talk (attended by the first author) at the University of Georgia by Erich Wittmann of the University of Dortmund who was motivated by ideas discussed in the 1990s within the UK Association of Teachers of Mathematics; see also [4, 5]. In this talk, concerned with the interplay between problem solving and preparation of mathematics teachers, the following problem was presented by Whittmann: *Given natural number N as the fifth term of a Fibonacci-like sequence, find all possible non-negative initial values.* In terms of cookies and plates, when $N = 8$ the first two plates have one and two cookies (as initial values), respectively; when $N = 11$, there are two options to put cookies on the first two plates: one and three cookies and four and one cookies, respectively. However, when $N = 11$ is the fourth term of a Fibonacci-like sequence, there are more than two ways to put cookies on the first two plates. It is this case that was used by the authors with 3^{rd} and 5^{th} graders. Ironically, as will be discussed below, the former group was more successful in solving the problem.

The second problem was included in earlier editions of the notable textbook for teacher candidates used in the past at the university where the authors work: *"If the 5 key on your calculator were broken, how could you do this problem: 458 + 548 + 354? Is there more than one way? Which way do you like the best? Why?"* [6] (p. 67). It has been observed that elementary teacher candidates, perhaps prompted by the penultimate question (and even when the problem is presented through the first question only), see this problem as a purely computational activity and enthusiastically demonstrate their multiple solutions through the sums (or differences) like 448 + 448 + 334 + 10 + 100 + 20 (or 468 + 648 + 374 – 10 – 100 – 20), and then deal with computations. (A possible computerization of such problems using *Wolfram Alpha* followed by the use of basic techniques of combinatorics known to elementary teacher candidates will be shown in Section 6 below). Their computational enthusiasm about this problem is unfortunate for it demonstrates the lack of conceptual thinking undeveloped in grade school where mathematics is often taught as a pure computation structured by the Pemdas (parentheses, exponents, multiplication, division, addition, subtraction) rules (https://study.com/learn/lesson/pemdas-rule-equation-examples.html, accessed on January 29, 2024). This was the main reason for the authors recent work in the upper elementary classrooms at a rural school in Upstate New York with minority student enrollment 97%. The school serves as a PDS (professional development school) site for SUNY Potsdam in the Holmes Group [7] sense. The group of third graders included six students (3 girls and 3 boys). The group of fifth graders included seven students (3 girls and 4 boys).

2 On the importance of motivation in the teaching of mathematics to children

Max Wertheimer, one of the founders of Gestalt psychology, argued that for many children "it makes a big difference whether or not there is some real sense in putting the problem at all" [8] (p. 273). He gave an example of a nine-year-old girl who was not successful in her studies at school. In particular, she was unable to solve simple problems requiring the use of basic arithmetic. However, when given a problem which grew out of a concrete situation with which she was familiar and the solution of which "was required by the situation, she encountered no unusual difficulty, frequently showing excellent sense" [*ibid*, pp. 273-274]. Benoit Mandelbrot, the author of fractal geometry, in a plenary lecture at the 7th International Congress on Mathematical Education in 1992, advised mathematics teacher educators, the main group of attendees at the Congress, regarding the need for motivation when teaching mathematics: "Motivate the students by that which is fascinating, and hope that the resulting enthusiasm will create sufficient momentum to move them through that which is no fun but is necessary" [9] (p. 86). Likewise, William James, an American educational psychologist of the 19^{th} century, advised future teachers as follows: "Any object not interesting in itself may become interesting through becoming associated with an object in which an interest already exists" [10] (p. 62). Another relevant instrument of motivation is the real-life focus of mathematical activities. According to David Hilbert, a notable German mathematician of the 19^{th}-20^{th} centuries, mathematics begins with posing problems in the context of concrete activities "suggested by the world of external phenomena" [11] (p. 440).

More recently, motivation was connected to some level of challenge that educational tasks should incorporate so that when students experience success with such tasks, they feel good about their learning [12, 13]. The problem with cookies was motivational in at least two aspects: engaging context and suitable challenge. Kids like cookies and although cookies were pictures, there still were concrete activities that were fascinating and external to the mathematics of counting, addition, and subtraction. The challenging aspect of the problems with cookies and the broken key was in the presence of more than one correct solution, something that is considered as path to creativity [14, 15]. The problem with the broken key, as will be shown below, was motivated by a concrete situation familiar to the fifth graders.

3 Cookies on plates (Grade 3)

3.1 Problem 1

If the fourth plate has 11 cookies, how many cookies can be put on the first and the second plates, so that the third plate has as many cookies as the first two plates and the second and the third plates have 11 cookies combined?

3.2 Discussion

The third graders worked on the problem without help and although they (collectively) missed one case, (1-5-6-11), the result is quite impressive. A few students working at a higher ability level searched for multiple possibilities where lower students (with some the second author worked with math intervention for two years) celebrated their success in finding just one solution. The third graders worked on these activities using their individual "plates". They did not ask if they had found all the solutions, and their answers/results were not posted on the board as they discovered the possibilities due to "copying" what has been already solved. One of the students who found multiple possibilities, was very quick to find his first combination and wanted to try and find another way. Encouraging him multiple times to "go for it" resulted in a self-posed problem with 12 cookies on the fourth plate solving which yielded four ways (out of five total) to put cookies on the first two plates.

The students realized that there is a difference between a problem with three plates and the problem with four plates, both in terms of mathematical complexity and things to recognize. Whereas in the case of three plates the order in which cookies are put on the first two plates does not matter, it does matter in the case of four plates. Because, inadvertently, knowing that addition is a commutative operation may lead one to believe that the order in which numbers are added does not affect the result, "it is important for students to understand when order matters" [16] (p. 201). The problem with cookies on more than three plates represents a context when order matters. It seemed to be early to talk about Fibonacci numbers but looking into the future, the problem with cookies on plates provided young children with enjoyable context associated with Fibonacci numbers, so that when time comes to study the famous numbers, the students would have enjoyable context to recollect.

Some students used the backward subtraction when solving Problem 1: when they placed cookies on the third plate, they were observed to "count up" from that plate to see how many they need on the second plate to have 11 cookies on the fourth plate. One of the subtraction strategies taught to third graders when they were in the first and the second grades through intervention was "count up to subtract" or "add on to subtract." Subtraction is a difficult concept for young children because counting backwards is not a natural process for them. So, as they see a problem for subtraction, i.e., in subtracting 7 from 11, instead of counting backwards from 11 to 7, they were taught to start at 7 and count upwards to 11 to get their answer, 4. Likewise, they started at 4 and counted upwards to 7 to get 3 cookies for the first plate. This is the strategy a few students used to engage in this activity, clearly using their previous knowledge. (see Figures 1 to 4)

4 Cookies on plates (Grade 5)

The 5^{th} graders were offered the same problem as the 3^{rd} graders (Problem 1, Section 3). Whereas the 5^{th} graders found only three solutions (Figures 5 to 7), their strategies were observed as more advanced. It was a few minutes that the fifth graders were able to figure out to work "Backwards" (subtract) from the 11 cookies to see what they could add up to equal the fourth plate from the third and second plate (using subtraction). They then worked to find what would

Figure 1 5 + 3 = 8 and 3 + 8 = 11

Figure 2 7 + 2 = 9 and 2 + 9 = 11

Figure 3 9 + 1 = 10 and 1 + 10 = 11

Figure 4 3 + 4 = 7 and 4 + 7 = 11

be needed to add to the second plate to equal the third plate (by placing what they needed to the first plate). In other words, the fifth graders were using the "well-ordering principle" [17] (p. 34). Namely, subtracting 10 from 11 yields 1 and the quadruple (9, 1, 10, 11); subtracting 9 from 11 yields 2 and the quadruple (7, 2, 9, 11); subtracting 8 from 11 yields 3 and the quadruple (5, 3, 8, 11); subtracting 7 from 11 yields 4 and the quadruple (3, 4, 7, 11); subtracting 6 from 11 yields 5 and the quadruple (1, 5, 6, 11); subtracting 5 from 11 yields 6 and one can see that this is the first time when the number of cookies on the third plate became smaller than that on the second plate. One can see that the above subtraction follows the system: from 10 to 9 to 8 to 7 to 6. For some reason, two cases, were missing, perhaps because fifth graders were not familiar with systematic reasoning. Yet, the very fact that the students were capable using this idea of working backwards without any instruction is quite impressive. It is worth noting that a similar problem with cookies was offered to teacher candidates enrolled in the course and only one candidate from about 50 used the "well-ordering principle", that is, worked backwards through subtraction in finding the number of cookies put on the first two plates.

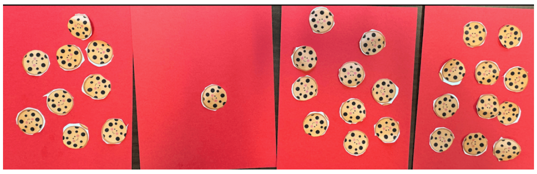


Figure 5 5^{th} Grade: 9 – 1 – 10 – 11

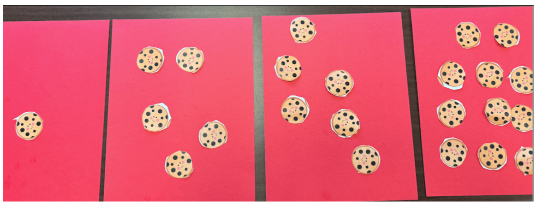


Figure 6 5^{th} Grade: 1 – 5 – 6 – 11

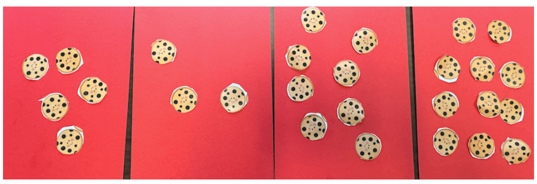


Figure 7 5^{th} Grade: 5 – 3 – 8 – 11

4.1 Remark

It is interesting to compare two problems solved by the fifth grades and third graders. In order to put 11 cookies on the 4^{th} plate following the "add the cookies on the last two plates" rule, one can note that the quadruple $(x, y, x + y, x + 2y)$ represents the quantities of cookies on four plates. To put 11 cookies on the fourth plate, one has to find the positive integer values of x and y such that $x + 2y = 11$. The last equation can be solved by noting that because $2y$ is always an even number and 11 is an odd number, the value of x must be an odd number smaller than 11. This observation yields $x \in \{1, 3, 5, 7, 9\}$ whence $y \in \{5, 4, 3, 2, 1\}$. At the same time, the problem with 11 cookies on the third plate solved by the third-grade students as a warm-up activity have the same number of solutions (when the commutative property of addition is not used) because the equation x + y = 11 has the same number of solutions as the equation $x + 2y = 11$, assuming $x \leq y$ in both cases. Another comparison of the two groups deals with the issue of missing quadruples. For example, the quadruple (1, 5, 6, 11) was missed by the third graders but found by the fifth graders. At the same time, the quadruples (3, 4, 7, 11) and (7, 2, 9, 11) were found by the third graders and missed by the fifth graders.

5 A calculator with a broken key (Grade 5)

In order to alter the status quo with the lack of conceptual thinking in grade school mathematics, or, alternatively, to show that young children are capable to think conceptually, the authors offered to seven fifth graders a modification of the original broken key problem [7] (p. 67) – having the addends 213, 342, 431 (not to have more than nine hundreds, tens, and ones in the sum in order to avoid unnecessary errors/challenges in tactile solutions) and the key 2 broken. The second author, assisted by teacher candidates, used drawn HTO place value charts with images of circles, squares, and triangles serving as ones, tens, and hundreds, allowing students to move the shapes (vertically but not horizontally) in order not to have two identical shapes as a face value. Then, the students were advised to move from visual to symbolic by replacing what they see by numbers and knowing that the sum stays the same, so they do not need to deal with computations over and over. The students were interested in creating their own tactile solutions to the problem and some were even posing new problems of that type. That is, mathematics was presented to the students as a game with certain rules the result of which, and not the game itself, can be described numerically. Knowing about successful use of conceptual thinking by young students does have positive impact on teacher candidates when the goal of education research is "to connect their methods experiences with Pre-K-12 classrooms" [3] (p. 36); that is, linking education theory to classroom practice.

5.1 Motivating the broken key problem

As a motivation for the broken key problem, the following situation was discussed by the second author with the fifth graders.

5.1.1 A food for thought

Recently, I tried to log in into my school computer, but the password I used was rejected. I was sure the password was correct, but each time I pressed correct keys on the keyboard, the password was rejected. What went wrong? Any ideas?

Responses of the participating students were recorded during a round table discussion. The responses were as follows.

Student A: *I must have spilled something sticky on my keyboard so the key got stuck and won't work. Teacher says no food by the computer!*

Student B: *My nails need to be shortened! Then I should try again and make sure I was pressing the right keys, my mom dials numbers wrong all the time because of her nails.*

Student C: *Shake the computer or hit the back of it (works all the time for our TV remote).*

Student D: *Wash my keyboard with a wipe.*

Student E: *Take it to the office.*

Student F *Call the tech guys to fix it.*

Student G: *Shrugged shoulders.*

The next questions during the discussion were as follows. The questions were aimed at distinguishing between letters and numerals. Whereas a letter may not have an alternative representation, a numeral may.

5.1.2 What letter do you think is very often used when typing?

Responses:

Student A: *Letter A, E, or O and Period*.

Teacher: *Let's pick one*.

Student C: *Response: "a"*.

5.1.3 What would you do if you realized that the letter 'a' key on your keyboard wasn't working ?

Responses:

Student B: *Tell the teacher*.

Student A: *Have the tech guy fix it*.

Student C: *Use voice to text typing*.

Student F: *Write it in Japanese because they use symbols then use Google translator to turn it to English*.

5.2 A simple problem

In order to move from the food for thought to mathematics, and to show the difference between letters and numerals, the following simple problem was offered to the fifth graders.

5.2.1 Problem 2

What if we want to find the sum 5 + 6 using a calculator but the key 6 is broken. Can the sum, unlike a password, be entered differently?

Figure 8 shows one student's response to Problem 2. An interesting observation is the absence of commutative property of addition in partitioning 11 into sums of other numbers and the use of only two addends. Although this simple problem was offered after the students did the problem about cookies on plates, it was discussed in order to show how the relations shown in Figure 8 can be used to decide the number of cookies on four plates. For example, the relation 8 + 3 = 11 suggests that when the 4^{th} plate has 11 cookies, the 3^{rd} plate has 8 cookies and the 2^{nd} plate has 3 cookies, thereby implying that the 1^{st} plate has 5 cookies (see Figures 1 and 7). Connections standard is one of the five Process Standards set by the National Council of Teachers of Mathematics demonstrating how "Building on connections can make mathematics a challenging, engaging, and exciting domain of study" [18] (p. 204). Common Core State Standards emphasize the importance of "the ability to *contextualize* ... in order to probe into the referents for the symbols involved" [1] (p. 6). Connecting Problem 1 to Problem 2 in the course can help instructors to alter some candidates' "own views about what it means to know and do mathematics ... believing that learning to teach is a matter of learning to explain procedures clearly" [2] (p. 34).

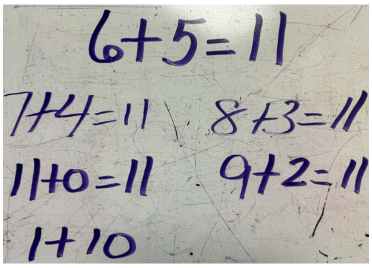


Figure 8 Solving Problem 2

5.3 Using a place value chart (Grade 5)

5.3.1 Problem 3

If the key 2 on your calculators were broken, how could you find the sum 213 + 342 + 431? What is this sum?

5.3.2 Discussion

The problem was presented by the second author on a whiteboard (Figure 9). Two students immediately made an observation that none of the shapes in the middle "tens place" has to be touched or rearranged. The students were using small dry erase boards to solve the problem and to find the sum. Figure 10 shows nine triangles, eight squares, and six circles. In other words, the sum is equal to 986. One comment about eight squares/rectangles is worth mentioning. The top square is drawn as a rectangle, visually very different from others representing tens. Usually, the larger the number one has to represent, the more space one needs to create a tactile representation of this number [19]. But when the space is the same, we see an attempt to have a ten being represented by a larger shape when same space is available for the representation of a larger number – the ten shrinks.

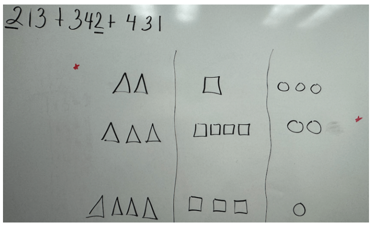


Figure 9 Presenting Problem 3 on the whiteboard

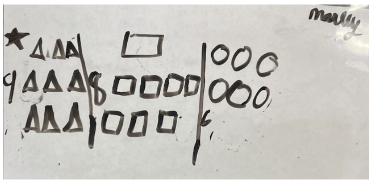


Figure 10 Recording the sum: 213 + 342 + 431 = 986

Figure 11 shows another representation of solution to Problem 3 – a fifth grader copied teacher's problem representation (Figure 9) and then showed the very process of dealing with the broken 2 key. The arrows include multiple ways of writing the sum of three-digit numbers without doing addition and not using the digit two as far as the rules of the HTO chart are not violated. And, once again, the sum of the three numbers is equal to 986.

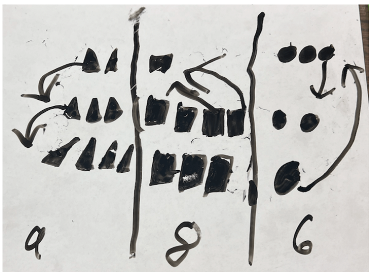


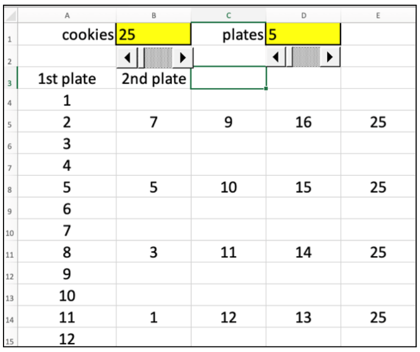
Figure 11 Solution of Problem 3 in action

6 Back to a teacher education classroom

This section is written to discuss what can be brought back to the classroom of elementary teacher candidates as a reflection on and an extension of the activities described in the previous sections of this paper. This discussion will include the use of technology such as a spreadsheet and *Wolfram Alpha*. Although young children did not ask an information type [20] question "how many?" regarding the problems with cookies on plates and a broken key on a calculator, teacher candidates might want or sometimes even need to know the number of solutions when offering, perhaps self-posed, problems with more than one correct solution to their own students.

In the course taught by the authors, a spreadsheet shown in Figure 12 is introduced in which one can vary the number of plates and the number of cookies on the last plate. The spreadsheet is programmed to display all possible answers to the cookies on plates problem. Such a spreadsheet is also included in the textbook used in the course (see [21], p. 87]). The programming of the spreadsheet is based on the Binet's formula for Fibonacci numbers and discussed in section 13.6 of the textbook. For example, the spreadsheet of Figure 12 shows that when 25 cookies are put on the 5^{th} plate, there are four ways to put cookies on the first two plates: (2 & 7), (5 & 5), (8 & 3), and (11& 1).

The problem with a broken key can also be discussed in a technological context using *Wolfram Alpha*. The tool has been used with elementary teacher candidates as described in detail in [22]. In the context of the broken key, *Wolfram Alpha* can be used through a combination of digital computation and formal reasoning involving such concepts of elementary school mathematics as additive decomposition of integers, the rule of product, and permutations of objects in a set. The problem, that was solved by 5th graders using a tactile approach, involved the sum 213 + 342 + 431 to be entered into a calculator without dialing the key 2. As shown in Figure 10, the sum of the above three addends (found by a 5th grader) has nine hundreds, eight tens and six ones that are represented through triangles, squares, and circles, respectively, as 9 = 3 + 3 + 3, 8 = 1 + 4 +3, and 6 = 3 + 3 + 0. This solution is one out of 16, 200 ways to enter the sum of the above there-digit numbers without using the (broken) key 2. To explain how the above five-digit number can be found, note that 9 = 3 + 3 + 3 is one way to partition 9 into a sum of three non-negative integers. The same can be said about the partitions 8 = 1 + 4 + 3 and 6 = 3 + 3 + 0. With this in mind, one can enter into the input box of *Wolfram Alpha* the command "solve over the integers x+y+z=6, x≥ y≥ z≥ 0" and the like command for 8 and 9. Figure 13 shows that the case of six (ones) provides seven partitions out of which the triples (3, 3, 0), (4, 1, 1), (5, 1, 0) and (6, 0, 0) do not include the number 2. The elements in the last four triples can be permuted (the concept of permutation is included in the course), respectively, in 3, 3, 6, and 3 ways, yielding the total of 15 permutations. The solution (by a 5th grader) displayed in Figure 10 shows one such permutation, (3, 3, 0) of the above first triple. Likewise, using *Wolfram Alpha*, one can conclude that in the case of tens, the triple (1, 4, 3), shown in Figure 10, is one permutation out of 27 total. In the case of hundreds, the triple (3, 3, 3), shown in Figure 10, is one permutation out of 40 total. That is, there are 40 ways to arrange nine triangles (hundreds) in three groups, 27 ways to arrange eight squares (tens) in three groups, and 15 ways to arrange six circles (ones) in three groups, no group having two shapes. By the rule of product (studied in the course), there are $40 \times 27 \times 15 = 16,200$ ways to enter the sum 213 + 342 + 431 into a calculator without dialing the key 2.



A	B	C	D	E
cookies	25	plates	5	
1st plate	2nd plate			
1				
2	7	9	16	25
3				
4				
5	5	10	15	25
6				
7				
8	3	11	14	25
9				
10				
11	1	12	13	25
12				

Figure 12 Cookies on plates spreadsheet

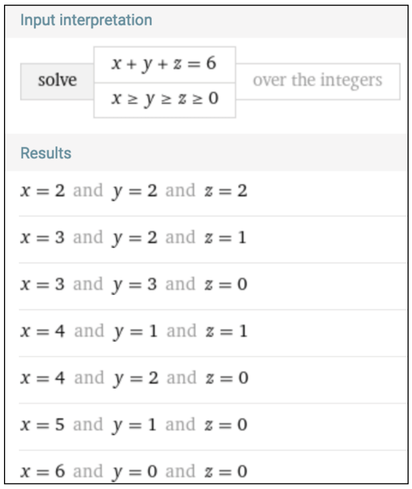


Figure 13 *Wolfram Alpha* as a three-dimensional partitioner

7 Conclusion

This paper stemmed from the authors' intent to follow a recommendation by one of the major agents of educational change in the Unites States [3] and connect two different groups of learners of mathematics: elementary teacher candidates working towards their master's degree in education and students from two upper elementary classrooms of a local school. The choice of the school was two-fold. It was a PDS site for the college of education that employs the authors. Also, the second author did intervention activities in the school and was familiar with many of the participating students, both 3rd and 5th graders. Not the small measure in the choice of the site for the authors' research was the school's 97% minority enrollment. By capitalizing on this enrollment number, the authors wanted to test their belief that a problem-solving pedagogy with its emphasis on productive thinking and hands-on mathematical activities is of the primary importance for one's success in the learning of mathematics, regardless of students' demographic. In the mid 20th century, Max Wertheimer and his collaborators in Gestalt psychology research [23] studied problem solving by young children in public schools of various parts of the Brooklyn borough of New York city in the United States. One such school was in a district "populated by recent immigrants who were minority group members and who were of the lower class" [24] (p. 62-63). The researchers then found that it is progressive pedagogy with its emphasis on critical thinking and experiential learning and not a student population that results in success in mathematical problem solving by students. This finding from the mid 20^{th} century is exactly what the authors were able to confirm at the end of the first quarter of the 21^{st} century by working with minority students in a rural school in upstate New York. Motivation to solve problems, high level of on-task behavior, and, most importantly, "desire to go on learning" [25] (p. 49) outside of the traditional curriculum demonstrated by young children in the context of the cookies on plates problem are valuable pieces of information that the authors have secured to share with elementary teacher candidates, some of whom will be doing student teaching at the same school. A worthy of sharing with the teacher candidates was the fact that 5^{th} graders, without any advice from the second author as a "more knowledgeable other" [26] were, unlike teacher candidates in a graduate program, using the "well-ordering principle" [17] (p. 34) when solving the cooking on plates problem.

The paper also demonstrated what other ideas can be shared with teacher candidates as a reflection on research done at the PDS site. Whereas young children did not use digital tools when solving problems, their interest in exploring ideas outside the traditional curriculum indicates the need for developing new curriculum materials. It is where technology can play an important role by assisting the candidates in problem posing, an activity described in [27] as "the agent of change within scientific paradigms" (p. vii). The spreadsheet shown in Figure 12 can be used by the candidates as such an agent. Furthermore, the use of *Wolfram Alpha* as a less known instrument of elementary mathematics teacher education can serve as a bridge between upper elementary and graduate teacher education classrooms. As noted in *Conference Board of the Mathematical Sciences* [2], elementary teacher candidates "must have opportunities to engage in the use of a variety of technological tools ... even if these tools are not the same ones they will eventually use with children" (p. 34).

Using information provided by *Wolfram Alpha*, the ideas of this paper can be extended to the secondary level of mathematics, both for high schoolers and their future teachers. The latter group can be provided with research-like activities leading to posing various problems of the broken key type. For example, as shown in Figure 13, when the number 6 is partitioned in three addends both the numbers 1 and 2 appear in three triples. When the number 8 is partitioned in three addends both the numbers 1 and 2 appear in four triples. Furthermore, both digits 1 and 2 appear in the sum 213 + 342 + 431 exactly two times. Yet, one can check that there are 9, 672 ways to enter the above sum into a calculator with the key 1 broken, a number much smaller in comparison with 16,200 when the key 2 is broken. And in the case of the digit 4 (if serving as the broken key) that also appears twice in the sum, this number is 25,080. Other sums can be used for secondary teacher candidates' independent research-like projects aimed at exploring the number of possibilities to enter the sums into calculators with different keys broken.

This kind of technology-enabled research in the context of school mathematics is an example of how hidden mathematics curriculum [28] associated with problems originally designed for the elementary level can be revealed to students at the secondary level as well as to their future teachers of mathematics. Furthermore, the extension of the broken key problem to the secondary level provides an opportunity to follow a recommendation that history of mathematics "can either be woven into existing mathematics courses [for prospective high school teachers] or be presented in a mathematics course of its own" [2] (p. 61). Indeed, the situation when one might expect the broken key 1 not to yield such a big difference in the number of possibilities to enter three three-digit numbers into calculators with either the key 1 or the key 2 broken is not uncommon in mathematics and well known in its rich history. It is this aspect of the number of permutations to be dependent on whether triples have or have not repeated elements that was behind a famous problem solved by Galileo Galilei (1564-1642) who was asked by a friend [29] (p. 4-5), alternatively, by "the Grand Duke of Tuscany, his benefactor" [30] (p. 2), as to why the number 10 appears more often than the number 9 when rolling three dice, although both numbers have six additive partitions in three unordered natural numbers not greater than six. Such "historical snippets" [31] (p. 214) have also a pedagogical value because they "help prospective teachers appreciate how hard these ideas can be for students who encounter them for the first time" [2] (p. 62).

Many other problems used in the course can be explored with young children conceptually in a tactile manner and likewise expanded to the secondary level with digital tools. Such perspective on the use of problems designed primarilly for elementary teacher candidates makes it possible to behold the problems through the lens of substantial learning environment [32] which can be used by mathematics educators bi-directionally *beyond* the problems' original purpose, both in grade school and at the secondary level. The use of tactile and digital technology, enhanced by historical perspectives, appropriately contributes to this environment. The authors believe that epistemological awareness of omnipresent nature of basic mathematical concepts has great potential to enhance a progressive pedagogy of the entire K-12 mathematics education.

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