RESEARCH ARTICLE

Mathematics Teachers as Applied Mathematicians: Advancing Teacher Education in the Age of Technology

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Abstract: Applied mathematics represents a blend of methods of pure mathematics and knowledge of problems from a field to which those methods can be applied toward the advancement of the field. Notably, the fields of engineering and life sciences have been significantly advanced by formulating their problems in the language of mathematics and using rigorous mathematical methods to enable intuitive thinking to be replaced by exact models to which formally proved mathematical propositions can be applied. This paper suggests that among the fields to which knowledge and methods of mathematics can be applied, especially in the age of technology which supports experimental problem solving, is mathematics teacher education. The paper builds on the ideas about mathematics as an experimental science that span from ancient to modern times. It provides three major illustrations of different levels of contextual and conceptual intricacy reflecting on the author's work as a mathematician-teacher educator. This conceptual reflection alludes to a case in point that seeing mathematics pedagogy through technology-enhanced lens can sometimes give way for the emergence of new mathematical knowledge through learners' unwitting entry into the substance of the discipline. Examples of such unexpected entries provided in the paper include the discovery of generalized Golden Ratios as strings of numbers of different lengths, Fibonacci-like polynomials stemming from a rearranged Pascal's triangle, and symmetrical vs. asymmetrical location of the roots of the polynomials depending on the sums of their coefficients.

Keywords: applied mathematics, teacher education, computational experiment, trial and error, early algebra

1 Introduction

It has been almost three decades since the first editorial to the *Journal of Mathematics Teacher Education* advised the readers that "knowledge domains of mathematics teacher education have no sharp demarcation lines and often involve fields seemingly unrelated, at first glance, to the education of mathematics teachers" [1, p. 1]. In the spirit of this advice, invoking the enactment of diverse viewpoints across related outlets, the present paper intends to reflect on theory and practice of mathematics teacher education through lens of the field of applied mathematics. In doing so, the author will attempt, by means of "reflecting on one's own *teaching*, ... to [show] how ... computers be used to arouse *didactical* understanding" [2, p. 148, italics in the original] of mathematics, something that is still considered as one of the major problems of mathematical education across grades [3,4]. The goal of this conceptual reflection is to put forward the idea of mathematics education being one of the provinces of applied mathematics. Consequently, this idea implies that mathematics teacher education can be advanced under the care of diverse applications of the subject matter to education.

Applied mathematics represents a blend of methods of pure mathematics and knowledge of problems from a field to which those methods can be applied toward the advancement of the field [5–11]. Within the past fifty years, the fields of engineering and life sciences have been significantly advanced by formulating their problems in the language of mathematics and using rigorous mathematical methods to enable intuitive thinking and approximate prototypes of real-life situations to be replaced by exact models to which formally proved mathematical propositions can be applied. For example, through the rigorous study of such models, the limiting structure of solutions to even simple non-linear difference equations describing oil well drilling [12] and/or population growth [13] had been found to be extremely complex. To a certain extent, theoretical advances in the context of real-life situations were due to the germane interplay, as

Felix Klein (1849-1925), the first president of the International Commission on Mathematical Instruction (ICMI), would have put it, between "approximation and precision mathematics" [14, p. 196]. These two types of mathematics were already established in antiquity when the work of Euclid in elementary geometry and number theory demonstrated "a sharp separation between pure and applied mathematics, which goes back to Plato and Aristotle" [15, p. 84], and was, as Ormell [6, p. 128] put it, "an aristocratic pursuit: the unmotivated, original eruption of pure reason as a force in human society." The motivation for precision mathematics "came on the scene with the development . . . conditioned in the 16^{th} and 17^{th} centuries by the new demands of technology and of manufacturing" [16, p. 38].

Over the centuries, this interplay between the applied (approximation) and the pure (precision) was exemplified through Archimedes' [17, p. 13] "mechanical method did not furnish an actual demonstration", Kepler's "without proper experiments I conclude nothing" (cited in [18, p. 50]), and Euler's [19] 'observations and quasi-experiments'. Due to the effectiveness of experimental approach to mathematics practiced by the geniuses of the discipline, in the second part of the 20^{th} century it was suggested that mathematics courses for non-mathematicians integrate a variety of laboratory experiments "illustrating the nature of analogy between physical happenings in different engineering areas" [7, p. 386]. Furthermore, turning mathematics education "into an experimental science . . . [to] follow a carefully correlated pattern of observation and speculation, the pattern so successfully employed by the physical and natural scientists" [20, p. 242], was seen as a way of improving mathematics teaching and learning through field-based research. More recently, mathematics itself has been seen by leading pure mathematicians as an experimental science [21, 22] which, in the context of teacher education often serves as a research-like mathematical experience for teacher candidates [23]. Proceedings from the above pedagogic ideas stemming from the cultural tradition and academic praxis of the millennia-old mathematics, this paper, based on the author's teaching and research experience, is written to suggest that among fields to which mathematics and its methods of investigation are applied, especially in the age of computing technology, is mathematical teacher education.

In particular, the paper will demonstrate that seeing the pedagogy of mathematics through the lens of technology-enhanced subject matter allows for not only the advancement of educational ideas, but it can give way for the emergence of new mathematical knowledge [24]. It was observed [25] that pragmatic uses of technology in a mathematics classroom have either direct or collateral effect on the epistemic development of students, whatever the grade level. Often, students' technological experimentation with mathematical ideas can prompt asking questions and motivate creative thinking that are consequential for the advancement of mathematical education as applied mathematics. It is the knowledge of mathematics by teachers, who "view technology as an expected and vital part of the classroom" [26, p. 126], that allows for recognizing significance in students of all ages unwitting entry into the substance of the discipline [24, 27, 28]. Along with examples from traditional contents, the paper will provide few, less generally known, examples of such unexpected entries leading to generalized Golden Ratios as strings of numbers of different lengths, Fibonacci-like polynomials stemming from a rearranged Pascal's triangle, and symmetrical *vs.* asymmetrical location of the roots of the polynomials depending on the sums of their coefficients (which add up to Fibonacci numbers).

2 Materials and Methods

Two types of materials have been used by the author when working on this conceptual paper. The first type is digital, the so-called "mathematical action technologies" [29, p. xi]. These technologies include computational knowledge engine *Wolfram Alpha* developed by Wolfram Research (www.woframalpha.com, accessed on 18 November 2025), an electronic spreadsheet used by the author as a tool for numeric modeling, and computer algebra system the *Graphing Calculator* produced by Pacific Tech [30] that supports the construction of graphs. The second type of materials used by the author included teaching and learning mathematics standards and recommendations for teaching used in North America [26, 31, 32]. The standards call for fostering mathematical reasoning in the technological paradigm and using computer-generated representations of concepts when solving mathematical problems.

Methods specific for mathematics teacher education used by the author as applications of the subject matter include computer-based mathematics education, standards-based mathematics, and problem solving. Those methods are conducive to presenting teacher candidates with "connections between seemingly unrelated concepts" [31, p. 56] that are revealed through computational applications. The university where the author has been preparing teacher candidates to teach mathematics is near Ontario province of Canada, and many of the author's students are

Canadians pursuing their master's degrees in education. In Canada, teacher candidates learn how to think computationally by "expressing problems in such a way that their solutions can be reached using computational steps and algorithms" [32, p. 513]. In the United States, many secondary mathematics teacher preparation programs offer courses "that include topics such as . . . difference equations, iteration and recursion" [31, p. 66]. These topics underpin several computational algorithms used in this paper. The paper is supported by the candidates' reflections and perspectives on the (university) learning and (school) teaching of mathematics solicited through various assignments of mathematics education courses.

3 Teaching mathematics through applications across grade levels

The first issue of the journal *Educational Studies in Mathematics* published papers delivered at an ICMI colloquium devoted to the theme of teaching mathematics "so as to be more useful" [33, p. 3]. Three of those papers are of special interest as they, while not dealing with mathematics education of teachers, can nonetheless be connected to the latter domain. At the primary level, as mentioned by Lyness [34], whereas a child "learns mathematics through experiencing some situation which is an application of mathematics" (p. 98), older students learn to make mathematics applicable through solving genuine, open-ended problems under the guidance and encouragement of their teacher who provides "help when necessary to those that are floundering" (p. 103). Responding to the question of what it might mean to apply mathematics in the classroom, a teacher candidate (one the author's students) acknowledged, "*I have learned many useful strategies and tools to ensure that I am prepared to teach students how to problem solve. I have developed not just confidence but also strategies and tools to accommodate their learning when they require additional support*." This comment clearly echoes the position expressed in [34].

In the paper by Griffiths [35] dealing with university teaching of applied mathematics to physicists, the emphasis is on how pure mathematicians (rather than theoretical physicists as some suggested) could teach the subject matter as an experimental science to the benefit of students of different abilities. As this perspective apparently confirms the ability of mathematicians to teach applied mathematics through experiments at the tertiary level, teacher candidates should be provided with the development of mathematical habits of mind to do same things through their studies [31]. In the words of one of the author's students, "*I do feel that I've developed important mathematical habits of mind and can use them in the classroom. These habits include looking for patterns, reasoning logically, making sense of problems, and persevering in solving them—all key skills that mathematicians use. Developing these habits has made me a more confident future teacher*." Teaching schoolchildren for pattern recognition and perseverance in problem solving [36] and "imparting scepticism, curiosity and a humane attitude . . . of a thoughtful craftsman." [35, p. 18] represents application of traditional mathematical skills to the modern-day education.

Pollak [37] argued for applied mathematics being extended to wide areas of engineering and natural sciences and suggested that the essence of teaching applications of mathematics is not in solving a given problem but rather thinking about a given situation. When mathematics teacher education regarded through lens of application, open-ended problems, due to their acceptance of new ideas, may be seen as a class of problems one can think about, apply different problem-solving methods, and spontaneously extend it to new queries. In the words of another teacher candidate, "*Allowing students to experiment with mathematical ideas allows them to explore different approaches and think outside the box. The ability to do so fosters their creative thinking to generate solutions beyond memorization and allows them to create new relationships and connections. If mistakes are made, this process allows them to explore, be flexible in their thinking and build up their confidence in questioning.*" Such perspective on teaching mathematics is very promising for it helps students consider the four components of STEM (science, technology, engineering, mathematics) as the integrated whole.

Applied mathematics conferences, Aplimat, sponsored by Slovak University of Technology in Bratislava, provided its forum to mathematics educators (e.g., [38–40]). In the referenced conference papers, one can find reports of research on such traditional educational topics as the use of digital tools [38], problem solving [39], and teaching and learning fractions [40]. This indicates that mathematics educators have been recognized by applied mathematicians as partners who work on applying their knowledge of mathematics and awareness of students' difficulties with the subject matter toward improving mathematics education. Furthermore, the university in Bratislava publishes the Journal of Applied Mathematics and Engineerings (Aplimat) having New Trends in Mathematics Education among its fields with the focus on the use of new technologies. This provides another evidence of the role of technology in support of teachers acting in the classroom as applied mathematicians. Nonetheless, the idea of regarding pre-college mathematics teachers as applied mathematicians appears not being discussed by either mathematicians or educators. Therefore, because the interplay between mathematics education and applied mathematics can be found in the literature associated with mathematical teaching in the digital era, mathematics teacher education is worth being considered through the lens of applied mathematics.

In the past, there was no agreement between mathematicians and educators about how to act successfully in the mathematics teacher education classroom. This situation was similar to the one described in [35] about the call for theoretical physicists to teach applied mathematics to physicists at the university level. Perhaps due to "the dominance of education by professional educators who may have stressed pedagogy at the expense of content" [41, p. 189], one of the author's students, enrolled in a secondary mathematics education course, through the student opinion of faculty instruction, suggested that "the course has to prepare us to teach mathematics without knowing mathematics". To counter this belief challenging the importance of content in teaching school mathematics, something that "stands on its own as a domain ... needed by teachers for their work" [42, p. 398], there has been a call for "a shift of stress of pedagogics to subject matter" [43, p. 68]. Recently, it was suggested that such shift requires training teachers of the subject matter, whatever the grade level, under the dictum of a higher standpoint [14, 15] emphasizing how "concepts arise starting from observations and how they can be verified in practice" [44, p. 34]. By the same token, when mathematical concepts turn into principles ready to be used in applications, because of the danger of students missing "the power of making use of the principles in the simple cases occurring in the applied sciences ... in teaching it is not only admissible, but absolutely necessary, to be less abstract at the start, to have constant regard to the applications ... to combine simplicity and clearness with sufficient mathematical rigour" [45, p. 46].

4 Mathematics teachers as applied mathematicians

Conference Board of the Mathematical Sciences, an umbrella organization consisting of twenty professional societies concerned, among other things, with mathematical preparation of teachers, suggests that future teachers should be encouraged to "develop the habits of mind of a mathematical thinker and problem-solver ... to work in ways characteristic of the discipline. For teachers, this is not only worthy, but necessary" [31, p. 19]. This suggestion implies that when teaching mathematics, teachers are expected to act in the classroom as mathematicians by applying their knowledge of the subject matter to education. It is in that sense that mathematics teachers act in the classroom as applied mathematicians. As noted by one of the teacher candidates, the author's student, "*I believe I have developed mathematical habits of mind, and I can use them in the classroom by acting as a mathematician when teaching students to solve problems. These habits involve flexible thinking, solving problems strategically and reasoning logically, all of which are central to how mathematicians approach challenges. By embodying these habits, I can model these skills for my students, encouraging them to think in the same way when solving problems.*"

In the true spirit of applied mathematics, mathematics teachers (including mathematicians working in teacher education programs) are expected to be familiar with problems specific to mathematics education and theories developed to deal with those problems. Among such problems and theories, in addition to those mentioned almost half a century ago by Freudenthal [2], are the relation between procedural and conceptual knowledge [46, 47], the unity of problem solving and posing [48], the development of creativity and diversity of thinking [27, 49], mathematical knowledge for teaching [42, 50], conceptual reciprocity between mathematics education and mathematics [51], and the appropriate use of technology [29, 52, 53]. In the sections that follow, using several illustrations of problem-solving situations, the above theories will be integrated with technology, forward looking mathematics learning standards, and notable recommendations for teaching the subject matter that span over the centuries.

5 Illustration 1: From chickens and rabbits to grimps and drimps

5.1 Thinking through trial and error

The first illustration of preparing teacher candidates to act as applied mathematicians deals with developing their experience of thinking about a problem as a situation [37] through the combination of conceptual and contextual lens. Such thinking, which often starts with trial and error and nowadays may be technology supported, allows a candidate to better understand how the answer might look like [54], why it does not look like the one expected, and then select an approach that best fits the solution [8]. Here is a well-known task, which can be found in many places in various formulations, and, most notably, was discussed by Freudenthal [2, p. 137]: "A farm with chickens and rabbits, 20 heads and 56 legs; how many chickens and how many rabbits?" Freudenthal explains how to solve this problem by schematizing (i.e., procedurally by using algebra), assuming that the reader understands how to solve it by insight: "I am sure you will solve it by insight" (ibid, p. 137). Insight is not easy to define, especially in the context of this problem which, instead of algebraically, can be solved by trial and error supported by the use of technology. It is by thinking through trial and error that a teacher candidate can develop insight, something that an applied mathematician often uses when thinking about a situation [8]. The first step on that route is to assign all 56 legs to rabbits (as 56 is divisible by 4 – note this observation is not a part of algebraic solution which requires contextual understanding of the situation, i.e., knowing the number of legs a rabbit and a chicken have; the problem becomes more complex if this knowledge is absent; that is, mathematics to be applied depends on the level of understanding of the system in question). The second step is, by decreasing the number of rabbits by one and thus increasing the number of chickens by two, to increase the number of heads by one until the number 20 is reached.

The spreadsheet of Figure 1 is designed to provide didactical understanding of the following process. Starting with 14 rabbits and zero chickens (that is, with 14 heads), six decreases of rabbits and six increases of pairs of chickens are required to end up with 20 heads, 8 rabbits and 12 chickens. Alternatively, one can start with 28 chickens (where 28 is one-half of 56) and zero rabbits (that is, with 28 heads) and make eight decreases of pairs of chickens and eight increases of rabbits to achieve the same result. As a result, a teacher can see two ways of thinking about problem solving that can be characterized by their applied character. Starting with rabbits (14 heads) allows one to arrive at 20 heads faster than starting with chickens (28 heads).

56 LEGS

RABBITS	CHICKENS	HEADS	CHICKENS	RABBITS	HEADS
14	0	14	28	0	28
13	2	15	26	1	27
12	4	16	24	2	26
11	6	17	22	3	25
10	8	18	20	4	24
9	10	19	18	5	23
8	12	20	16	6	22
7	14	21	14	7	21
6	16	22	12	8	20
5	18	23	10	9	19
4	20	24	8	10	18
3	22	25	6	11	17
2	24	26	4	12	16
1	26	27	2	13	15
0	28	28	0	14	14

Figure 1 Trial and error supported by spreadsheet computing

While this difference in the number of trial-and-error steps is not significant (especially when technology does the counting), acting as an applied mathematician a teacher can show both methods to point out that one of them takes less time to arrive at the solution. In the context of spreadsheet modeling, didactical understanding stressed by Freudenthal [2] in connection with the use of computers in mathematics education can be attained by altering problem's data through the discussion of the efficiency of the trial-and-error method. Had we changed the number of heads to 26, the second solution (starting with chickens) would be more than 80% more efficient than the first solution (starting with rabbits). This enables the development of a rule suggesting, in the spirit of applied mathematics, which strategy to apply and when: the smaller the number of heads, the more likely the first solution is more effective, at least when using trial and error.

5.2 Developing insight through trial and error

How can one use this trial-and-error approach to develop insight necessary to program a spreadsheet of Figure 1? Perhaps insight to be developed can be described as realization that in terms of legs, one rabbit is equal (equivalent) to two chickens and, therefore, the exchange of a rabbit for two chickens yields the decrease in one head. Such insight, when demonstrated by a teacher candidate, indicates "the habits of mind of a mathematical thinker" [31, p. 19] who, by solving a procedural task conceptually, reveals how one can use "knowledge acquired by insight" [2, p. 137]. Furthermore, knowledge developed by equating two chickens to one rabbit in the domain of legs impels didactical understanding needed to program the spreadsheet of Figure 1 by decreasing the numbers in columns B and G by one and two, respectively, and increasing the numbers in columns C and H by two and one, respectively. One can see that in the context of the appropriate use of technology [29, 52, 53] didactical understanding may be of two types: one that emerges through the creation of modeling data and another one that emerges through the analysis of the data. That is, one can distinguish between pre-technology and post-technology didactical understandings that emerge and can be attained in the context of using a spreadsheet or any other educational instrument, for that matter.

5.3 Going beyond the farm

A teacher who acts as a mathematician in the classroom and, in doing so, connects real-life problems to mathematical methods available [8], would benefit from having the following citations in their tool kit of knowledge for teaching [42, 50]: "A farmer who seeks the rectangle of maximum area with given perimeter might, after finding the answer to his question, turn to gardening, but a mathematician who obtains such a neat result would not stop there" [55, p. 133] and "nothing happened in this world in which some reason of maximum or minimum would not come to light" (Euler, cited in [56, p. 121]). Indeed, a mathematician, for whom problem solving and problem posing are two sides of the same coin [48], is likely to pose the following questions: What is the largest/smallest number of legs among 20 chickens and rabbits? Is it possible, by changing the number of heads, to have twice as many chickens as rabbits?

Spreadsheet modeling makes it possible to pose new problems by changing numbers in the domains of legs and heads. As was mentioned above, didactical understanding includes the analysis of computer-generated data as a means of posing new problems. To answer the first of the above two questions, the graphical representation of the problem (Figure 2) allows one to see the largest/smallest number of rabbits in the case of 20 heads. As the graphs to be plotted require the use of algebra (i.e., schematizing), didactical understanding, attained by programming and analyzing the spreadsheet, can be utilized to construct two-variable linear equations about the number of heads of rabbits and chickens. Once the graphs have been plotted, they can be used for developing a post-technology didactical understanding of a higher order.

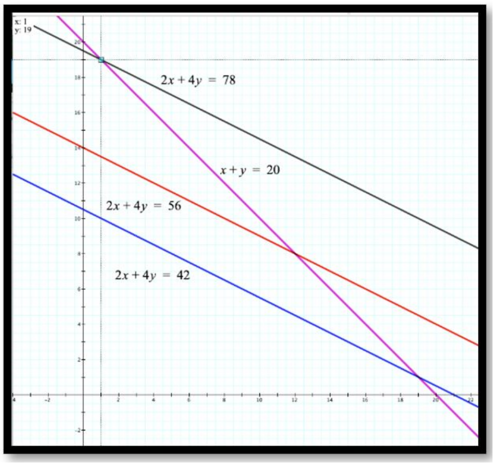

Figure 2 Graphical representation of the farm problem

However, the graph of the line $x + y = 20$ does not allow one to see easily the lattice points (and their coordinates) through which the line passes. Such values can be generated by Wolfram Alpha using knowledge of the largest/smallest number of rabbits shown by the graphical representation. One can see that, in general, numbers generated by Wolfram Alpha (Figure 3) do not repeat numbers generated by the spreadsheet (Figure 1). Nonetheless, some numbers are repeated. Recognizing repetitions is another strategy aimed at developing didactical understanding when using technology. Numbers presented by Wolfram Alpha can also be used to pose new problems thus using problem posing "to stimulate retention of insight, in particular in the process of schematizing . . . [by] having the learner reflect on his learning process" [2, p.

141]. In particular, these numbers confirm, what one can see on the graphs: the elements of the pairs (1,19) and (19,1) not only have same sum, but they provide, respectively, the largest and the smallest numbers of legs among chickens and rabbits.

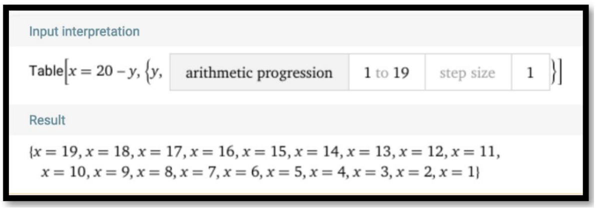

Figure 3 Using Wolfram Alpha in conjunction with the Graphing Calculator

5.4 Abstraction as a route to new contexts

The farm problem assumes that a problem solver has contextual understanding of the situation that is critical for solution. The problem may not be considered contextually coherent [57] for the one who is not familiar with the basic biology of the creatures involved. That is, if one does not know how many legs each creature has, he/she would have to make various assumptions by considering a farm inhabited by 20 artificial creatures of two types, called, for example, drimps and grimps [58, p. 91], with unknown number of legs each of them has, with 56 legs in all. This makes the problem while solvable yet more complex, be it an algebraic or a conceptual solution. This is how a mathematician might approach an applied problem by seeing it in more general terms to allow for thinking mathematically in the case of unfamiliar situation.

From the point of view of education, the reason for the abstraction is at least three-fold. First, it prepares younger students to deal with unknowns using trial and error. Second, the transition from "real life" to abstraction shows how context may include hidden information needed to solve a problem, yet this information might be outside the "cultural milieu" [20, p. 241] of some students. Third, the abstraction makes it possible to introduce a problem with more than one correct answer. Already in the case of 8 drimps and 12 grimps with 56 legs among them, there are two solutions: (4 legs, 2 legs; the special case of rabbits and chickens) and (1 leg, 4 legs) as $8 \times 4 + 12 \times 2 = 8 \times 1 + 12 \times 4 = 56$. In other words, the equation $8x + 12y = 56$ has exactly two positive integer solutions: (4, 2) and (1, 4). At the same time, whereas the equation $4x + 16y = 56$ (describing 4 drimps and 16 grimps) has exactly three positive integer solutions: (10, 1), (6, 2), and (2, 3), the equation $5x + 15y = 56$ has no positive integer solutions. One can use a spreadsheet to find the total of 26 combinations of 20 drimps and grimps having the total of 56 legs. This spreadsheet is shown in Figure 4 and it requires the circular reference as a special programming technique [57, p. 210] not discussed here for the sake of brevity.

	A	B	C	D	E	F	G	H	I	J
1	DIE 1	DIE 2		DIE 1/DIE 2		MARK INTEGER RATIO BY 1		COUNT 1's		EXP. PROB. INT. RATIO
2	2	3		0.6666667		0		788		0.394
3	6	6		1		1				
4	2	5		0.4		0				
5	4	1		4		1				
6	4	4		1		1				
7	5	4		1.25		0				
8	5	1		5		1				
9	2	2		1		1				
10	6	2		3		1				
11	5	2		2.5		0				
12	6	3		2		1				
13	2	4		0.5		0				
1998	5	6		0.8333333		0				
1999	5	5		1		1				
2000	6	1		6		1				
2001	3	3		1		1				
2002										

Figure 4 Exploring possibilities through a computational experiment

Transition from rabbits and chickens to drimps and grimps represents recognition that the farm problem can be modified to allow for a solution which is not dependent on the specific type of creatures. Whereas a purely algebraic solution is still possible and provides room for conceptual shortcuts like in the case of a two-variable equation $5x + 15y = 56$ where the absence of integer solutions is due to 56 not divisible by 5, knowledge of technology makes it possible, while still keeping the type of creatures unknown, to alter the total number of heads and legs. Finally, working with the spreadsheet enhances didactical understanding of the extended farm problem allowing for posing problems of the same structure yet related to different contexts. This is what applied mathematicians do: develop methods that match mathematical structure of a real-life problem and then expand the methods to cover other seemingly unrelated contexts. When a mathematics teacher acts in that way, he/she applies knowledge of mathematics to education.

Finally, note that with the smaller number of heads and legs, the extended farm problem can be solved by trial and error by drawing artificial creatures using the drawing features of MS Word. As mentioned by one of the author's students, "*When I was working one-on-one with a student in grade 6 who have modifications in their mathematic instruction and was learning at a grade 3 level of mathematic, using visuals and drawing to encourage them to show their work and see the mathematics displayed before coming up with their solution allowed them to be more confidents in their ability to solve mathematic problems as a student. For example, thinking about the drimps and grimps is an excellent example of how students can draw this on their own to work through the number of legs they have to ensure they can see the math laid out on the page and be confident in their answers*." This pre-student teacher's comment echoes a thought-provoking conviction that "the most abstract mathematics is without doubt the most flexible" [33, p. 5]. Indeed, the context of artificial creatures naturally motivates the integration of drawing and thinking under the flexible conditions of the uncertainty of early algebra.

6 Illustration 2: Applying mathematics to the assessment of students' work

6.1 A teacher in search for possibilities

Once upon a time, first graders were rolling two six-sided dice and recording the resulting sum on two faces. Just as the accuracy of computations provided by computers must be triangulated [59], especially when their "unreliability sometimes seems to be the norm" [60, p. 1400], the correct use of a calculator by a first grader depends on which button the child presses to carry out calculations. Indeed, the " $\div$ " button looks like the " $+$ " button on a tiny calculator. The teacher, seeing the answer smaller than one, without knowing numbers entered by the child, understood that, most likely, the child did division instead of addition, gave the child a larger calculator and reminded what the child is supposed to do.

The teacher realized that the " $\div$ " button can also be used when the first die has a larger number than the second die to have a non-integer result greater than one (e.g., 4/3 or 5/2). In that way, if mathematics is understood "as a discipline which we use to … explore possibilities" [6, p. 125], the teacher wanted to explore the chances for such outcome not to happen and proceeded acting as an applied mathematician by applying basic knowledge of mathematics to the assessment of an educational situation with the accidental use of the " $\div$ " key. The teacher knows that when the integer on the calculator's display is greater than one, the result may not necessarily indicate the use of the wrong key by the child. All this knowledge of arithmetic along with some basic counting techniques of probability theory was applied by the teacher to exploring possibilities through the assessment of the child's work and shared with the author through a final project.

There are six outcomes 1, 2, 3, 4, 5, 6 on the first die and, independently, same outcomes on the second die. The chances to have an integer quotient on the calculator's display when accidentally pressing the wrong button are: $\frac{1}{6}$ when the second die rolls 1 (with probability 1/6) regardless of how the first die rolls; $\frac{1}{12} = \frac{1}{6} \times \frac{1}{2}$ when the first die rolls 6 (with probability 1/6) and the second die rolls either 2, 3, or 6 (with probability 1/2); $\frac{1}{36} = \frac{1}{6} \times \frac{1}{6}$ when both dice roll 5; $\frac{1}{18} = \frac{1}{6} \times \frac{1}{3}$ when the first die rolls 4 (with probability 1/6) and the second die rolls either 2 or 4 (with probability 1/3); $\frac{1}{36} = \frac{1}{6} \times \frac{1}{6}$ when both dice roll 3; $\frac{1}{36} = \frac{1}{6} \times \frac{1}{6}$ when both dice roll 2. Thus, the (theoretical) probability to have an integer quotient is the sum $\frac{1}{6} + \frac{1}{12} + \frac{1}{36} + \frac{1}{18} + \frac{1}{36} + \frac{1}{36} = \frac{7}{18} \cong 0.39$. This result was confirmed experimentally by using a spreadsheet (Figure 4) which randomly generates 2000 pairs of the first six integers (columns A and B), computes their corresponding ratios (column D), marks integer ratios with 1's (column F), count 1's (cell H2), and computes an experimental probability (cell J2). This episode shows the application of mathematics and technology in assessment when thinking, in the spirit of Pollak [37], about child's accidental error in using a calculator as a situation.

6.2 Dealing with Einstellung effect

Often, when applying mathematics in context, applied mathematicians use their knowledge of a system and apply methods that are applicable to this specific system but lack generality and in the case of other systems provide erroneous results. Situation of that type in known in psychology as Einstellung effect [61] – one's tendency (mindset) to use a previously learned workable strategy in situations that either can be resolved more efficiently through a different approach or to which the strategy is not applicable at all. In order to deal with Einstellung effect in the context of assessment, a teacher must be able to recognize an error in student's work and explain it by referencing to Einstellung effect. Overcoming the Einstellung effect in problem solving indicates "that it is possible to change students' views from a fixed mind-set to a growth mind-set in ways that encourage them to persevere in learning mathematics" [31, p. 9].

For example, the trigonometric equation $sin^2x - sin\ x\ cos\ x = 0$, by dividing its both sides by cos^2x, can be reduced to a quadratic equation about witan^2xthout loss of solutions because $x = \frac{\pi}{2}$ does not satisfy the original equation. At the same time, exemplifying the Einstellung effect, dividing both sides of the equation $cos^2x - sin\ x\ cos\ x = 0$ by cos^2x results in the loss of solutions as $x = \frac{\pi}{2}$ does satisfy the latter equation (Figure 5: left and right, respectively; $y\prime$ is a notation, not a derivative). That is, the correctness of the strategy of dividing both sides of a homogeneous trigonometric equation of the second order $A\ sin^2x+B\ sinx\ cos\ x+Ccos^2x = 0$ by cos^2x depends on the coefficients of the equation. At the same time, when $B^2 - 4AC < 0$, the last equation does not have real solutions and the issue of dividing its both sides by cos^2x is irrelevant. This matter is also important in applications of mathematics: a problem-solving method applied to a system depending on parameters is expected not only to be stable regarding the change of the parameters involved but be also relevant to begin with [8]. A teacher, who informs a student that strategies working in a special case may not work in other cases that differ by the value of parameters or be immaterial as a strategy (as far as Einstellung effect in problem solving is concerned), acts as an applied mathematician dealing with the development of mathematical methods applicable (or be pertinent) to a class of systems rather than to a special case.

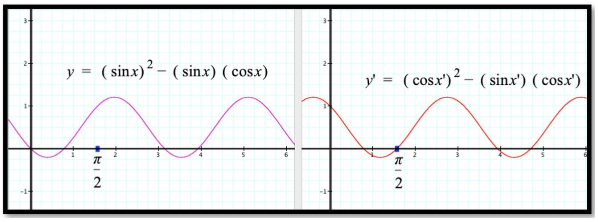

Figure 5 $y(\frac{\pi}{2}) \neq 0$; right: $y'(\frac{\pi}{2}) = 0$

6.3 Numerical evidence, pattern recognition, and the Internet

Mathematicians, both pure and applied, know well that "the first and oldest problems in every branch of mathematics spring from experience and are suggested by the world of external phenomena" [62, p. 440]. Having in mind such world as genesis of mathematics, the author once asked elementary teacher candidates to find out how many ways can one paint one, two, three, and four storied buildings so that no two consecutive stories are painted and unpainted buildings are included in the total count. One of the candidates correctly found (Figure 6) two ways for one-story buildings, three ways for two-story buildings, five ways for three-story buildings, eight ways for four-story buildings, and entered the numbers 2, 3, 5, 8 into the input box of Wolfram Alpha to get the continuation for this sequence. The tool responded with the numbers 12, 17, 23, 30 and provided the closed-form formula $a_n = \frac{1}{2}\left(n^2 - n + 4\right),\ n = 1,\ 2,\ 3,\ \ldots$ Yet, the goal of the task was to introduce Fibonacci number sequence by recognizing the relations 5 = 2 + 3, 8 = 3 + 5, and therefore the fifth term should be 13 = 5 + 8 and not 12. By encouraging behavior of an applied mathematician in assessing their own work, the candidate was asked to check the result provided by a digital tool by intuitively connecting two previous cases through painting a five-storied building in two steps. The first step is to paint the fifth floor implying that the fourth floor will not be painted leaving three floors to deal with, something that was already done. The second step is to leave the fifth floor unpainted, implying that the other four floors must be painted in a way it was already done. This kind of reasoning, shown in Figure 7, is akin to the act of sagacity defined by Aristotle as "hitting by guess upon the essential connection in an inappreciable time" (cited in [63, p. 58]).

Now, entering the numbers 2, 3, 5, 8, 13 into the input box of Wolfram Alpha provides continuation 21, 34, 55, 89. These numbers can be verified in context by recognizing that the equality 21 = 8 + 13 means the two step process of painting a six-story building: first, painting the first four floors in 8 ways, and, leaving the fifth floor unpainted, paint the sixth floor; second, painting the first five floors in 13 ways leaving the sixth floor unpainted. In terms of applied mathematics, this example illustrates how "results are tested ... by consistency and agreement with observations" [5, p. 876] and how "simple rudimentary methods [of coloring] are replaced by more sophisticated general methods [of reasoning by recursion]" [34, p. 103]. At the same time,

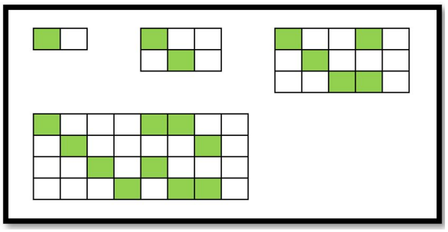

Figure 6 Hands-on finding the numbers 2, 3, 5, 8

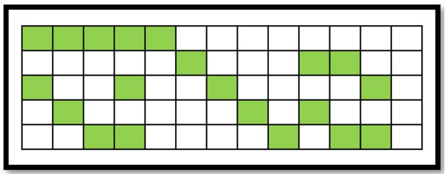

Figure 7 Verifying digital result using recursive reasoning

the result is based on limited numerical evidence and like experimental probability of head among ten tosses of a fair coin may be far away from 1/2 as the theoretical probability, limited numerical evidence for an unknown system may not be enough for a correct algebraic generalization.

In another instance, elementary teacher candidates were shown the arrangement of cookies on the four plates (Figure 8) and asked how many cookies one would put on the fifth plate based on the pattern they see. One of the candidates, familiar with Wolfram Alpha, entered the numbers 1, 3, 4, 7 into the input box of the tool and received the following continuation 6, 12, 8, 15, which, once again, did not represent what was expected if one follows the rule of a Fibonacci-like sequence. The tool also offered the general form $\sigma_1(n)$ with which the teacher candidate was not familiar. This case with cookies and Wolfram Alpha was even more complicated than the case with buildings. Indeed, the number of painted buildings was represented by an increasing sequence that can be connected to triangular numbers through the formula $\frac{1}{2}\left(n^2-n+4\right)=\frac{(n-1)n}{2}+2$, yet the function was no$\sigma_1(n)$t presented in a closed form. A curious learner can find out that repres$\sigma_1(n)$ents the sum of divisors of an integer *n* and the sequence 1, 3, 4, 7, 6 is due to

$$\sigma_1(1)=1,\ \sigma_1(2)=1+2=3,\ \sigma_1(3)=1+3=4,\ \sigma_1(4)=1+2+4=7,\ \sigma_1(5)=1+5=6.$$

How can a teacher didactically enhance the visual representation of objects shown in Figure 8 to help learners of mathematics recognize the recursive development of the number of cookies on plates? Figure 9 shows a possible representational modification that demonstrates Fibonacci recursion starting from the third plate. Once again, hands-on and visual strategies can serve as applied instruments aimed at motivating insight by means of allowing for "the most abstract mathematics taught within the most concrete context" [2, p. 144].

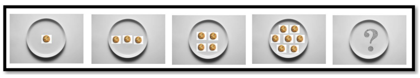

Figure 8 What pattern do cookies form?

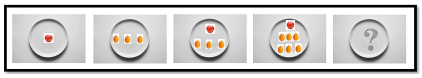

Figure 9 Representational modification to enhance visual comprehension

7 Illustration 3: From student's finding to Pascal's triangle to Fibonacci-like polynomials

7.1 Posing a challenge for a "more knowledgeable other" in the digital era

Application of mathematics to education in the context of teacher preparation sometimes results from the need to address students' unexpected findings when exploring mathematics with computers. Just as an engineer may discover a physical phenomenon explanation of which requires application of mathematical methods that do not exist and must be developed, a teacher candidate may accidentally come to recognize a computer-generated numeric pattern that contradicts to what was expected to be observed. In that case, the candidate would seek explanation from the instructor hoping that the "more knowledgeable other" is able to clarify. But the finding stemming from a computational experiment may be new for the instructor as well. This section is written to share just that case. As described in detail in [24], a secondary mathematics teacher candidate while using a spreadsheet reported to the author the following observation: in the parametric linear difference equation

$$f_{n+1}=af_n+bf_{n-1},\ f_0=f_1=1, \tag{1}$$

when $a=2$ *and* $b=-4$, the ratio $\frac{f_{n+1}}{f_n}$ does not have a number as a limiting value (like in the case of the Golden Ratio when $a = b = 1$) but, instead, expands into a string of numbers that repeats over and over as the value of n increases.

The illustration of this section shares the author's presentation of this story to secondary teachers who, when acting as applied mathematicians in their own classroom, are expected to "have the intellectual courage and mathematical disposition to not reject the challenge but to investigate the proposed idea, applying their own critical thinking and using all available resources" [26, p. 9]. The illustration does not start with modeling equation (1) within a spreadsheet, something that the teacher candidate, by "playing" with parameters, did. Rather, it demonstrates how theory required to address the challenge posed by the candidate was developed and communicated to future secondary teachers as an example of applying mathematics to education.

7.2 Rearrangement of entries of Pascal's triangle

The elements of Pascal's triangle known as binomial coefficients and shown in Figure 10 can provide many interesting and less generally known entries into discrete mathematics, especially when teachers have developed mathematical habits of mind and are trained in the appropriate use of technology. Those entries may be of applied nature when the results are provided by the use of technology and can serve as a motivation for the development of theoretical justification of technological evidence. But it is recognition of this evidence that requires mathematics teachers to possess mathematical habits of mind. As mentioned by one of the teacher candidates, the author's student, "*I was provided with opportunities to develop mathematical habits of mind that will carry through to my teaching career. I feel that I have developed such habits that can be used in the classroom and that will allow me to act as a mathematician when teaching students to solve problems. Furthermore, I believe in the importance of giving students the tools required to solve problems and develop their own mathematical habits of mind.*"

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

Figure 10 Pascal's triangle is symmetrical about central binomial coefficients

With such habits of mind, one can show how symmetry of those elements about the vertical altitude comprised of the numbers 1, 2, 6, 20 (known to mathematicians as central binomial coefficients) can be turned into an asymmetry of their rearrangement when the top-right/bottom-left diagonal with units forms the first column, the diagonal with natural numbers forms the second column shifted down about the first one by two rows, the diagonal with triangular numbers becomes the third column shifted down about the second one by two rows, the diagonal with triangular pyramidal numbers forms the fourth column shifted down about the third one by two rows, and so on (Figure 11). In that way, Pascal's triangle the elements of which form an evolving equilateral triangle is modified to form a (non-isosceles) right triangular staircase with each step comprised of two rows.

1					
1					
1	1				
1	2				
1	3	1			
1	4	3			
1	5	6	1		
1	6	10	4		
1	7	15	10	1	
1	8	21	20	5	
1	9	28	35	15	1
1	10	36	56	35	6

Figure 11 Turning diagonals of Pascal's triangle into columns yields asymmetry

Whereas it is well known that the sums of numbers in each pair of rows forming a step are consecutive Fibonacci numbers, if those numbers are used as coefficients of one-variable polynomials, then the polynomials have only real roots. These polynomials are called Fibonacci-like polynomials [24]. Furthermore, all the roots of the polynomials belong to the interval (– 4, 0) and alternate their positions within the interval from symmetrical to asymmetrical location. For example, consider the fourth step from the top (Figure 12) and the corresponding polynomials $x^3 + 5x^2 + 6x + 1$ and $x^3 + 6x^2 + 10x + 4$, the graphs of which are shown in the right- and left-hand sides of Figure 12. Both polynomials have their roots within the interval (-4, 0), yet the latter polynomial has symmetrical location of its roots, and the former polynomial has asymmetrical location of its roots within the interval. Such observations when made by using technology demonstrate applied character of mathematical entities and structures.

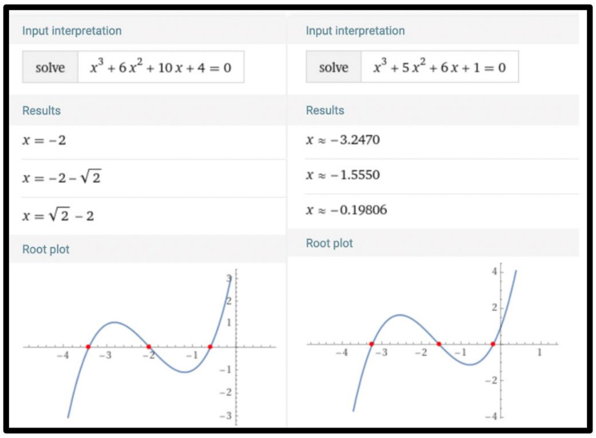

Figure 12 Fibonacci-like polynomials of degree three graphed by Wolfram Alpha

7.3 Connecting Fibonacci-like polynomials to a parametric difference equation

One can further use computer graphing to see that when the far-right element of the step (i.e., the free term of the corresponding Fibonacci-like polynomial) is equal to one, the location of roots is asymmetrical; when the far-right element of the step (i.e., the free term of the corresponding Fibonacci-like polynomial) is greater than one, then the location of roots is symmetrical. If x is replaced by $\frac{a^2}{b}$, equating so modified Fibonacci-like polynomials to zero yields two-variable equations $a^6 + 5a^4b + 6a^2b^2 + b^3 = 0$ and $a^6 + 6a^4b + 10a^2b^2 + 4b^3 = 0$ the graphs of which are parabolas (Figure 13) sharing the origin as their common vertex in the plane (a, b) and all six located inside the parabola $a^2 + 4b = 0$. This parabola defines the endpoints of the interval (-4, 0). Indeed, to satisfy the inequality $a^2 + 4b < 0$ (defining the space inside the parabola), one divides both sides of the inequality by (negative) b and, by returning to the original $x = \frac{a^2}{b}$, gets $0 > x = \frac{a^2}{b} > -4$.

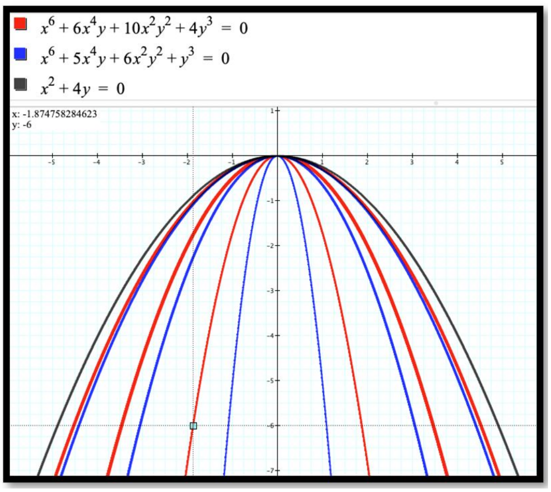

Figure 13 Two two-variable polynomials disintegrate in six parabolas

Note that the transition from one-variable Fibonacci-like polynomials to two-variable (parabola defining) polynomials is not intuitive and it stems from experience of dealing with applied problems in which one frequently parametrizes in order to unpack information hidden in an original representation by recourse to its geometric interpretation. That is how mathematical ideas and their representations develop: moving from Pascal's triangle, the entries of which are the outcomes of fair coin tossing, to its number-intact modification used to provide coefficients of polynomials with all, closely located, real roots each of which defines a parabola located inside the parabola defined by the length of the interval that coincides with the number of outcomes of tossing two coins and includes all the roots whatever the degree of a polynomial.

Furthermore, when a and b are used as coefficients (parameters) of difference equation (1) (when $a = b = 1$ the equation defines Fibonacci numbers through Binet's formula comprised of the (real) roots of the characteristic equation $\lambda^2 - a\lambda - b = 0$ which does not have real roots when $a^2 + 4b < 0$ – the most interesting case in the context of Fibonacci-like polynomials), setting $\frac{f_{n+1}}{f_n} = g_{n+1}$, linear difference equation (1) turns into a non-linear difference equation.

$$g_{n+1} = a + \frac{b}{g_n},\ g_1 = 1 \tag{2}$$

Now, selecting any point on any of the six parabolas of Figure 13 (setting $a = x$ and $b = y$ to satisfy the notation of the Graphing Calculator), for example, selecting $(a,\ b) = (-1.874758284623, -6)$, and iterating equation (2) within a spreadsheet (Figure 14), shows g_n forming an 8-cycle. The value of a is taken with 12 digits after the decimal point to ensure the accuracy of iterations forming cycles. By plugging $a = 2$, $b = -4$ into equation (2) yields a three-cycle $(1,\ -2,\ 4)$ discovered, as was mentioned above, by a teacher candidate. This unwitting entry into the fabric of mathematics represented an example of collateral creativity [27] made possible by the use of technology that called for both mathematical and didactical understanding of the emerging mathematical phenomenon [25].

	A	B	C	D
1	a	b	g(n+1)=a+b/g(n)	
2	-1.874758284623	-6	1	
3			-7.874758285	
4			-1.112830136	
5			3.516900159	
6			-3.580806147	
7			-0.199157944	
8			28.25208416	
9			-2.087131997	
10			1	
11			-7.874758285	
12			-1.112830136	
13			3.516900159	
14			-3.580806147	
15			-0.199157944	
16			28.25208416	
17			-2.087131997	
18			1	
19			-7.874758285	

Figure 14 Roots of Fibonacci-like polynomials as generators of cycles

Other Fibonacci-like polynomials can be modified (replacing x by $\frac{a^2}{b}$) and then equated to zero to construct new parabolas (inside the parabola $a^2 + 4b = 0$) the points of which can be used to form cycles of other lengths serving as generalized Golden Ratios [24]. The joint use of Wolfram Alpha, the Graphing Calculator and a spreadsheet provides the context of Fibonacci-like polynomials with computational triangulation [59], something that can be described as an application of mathematics to the problems of mathematics education in the digital era, including reliability of results provided by digital tools.

8 Conclusion

The paper was written to discuss the idea of mathematical education being a branch of applied mathematics. Just as mathematicians use their knowledge of mathematics when solving real-life problems "suggested by the world of external phenomena" [62, p. 440], the current focus on teaching mathematics, 'so as to be useful' [33], through technology-enhanced problem solving in context motivates and encourages teachers to apply their mathematical content knowledge

[42, 50] as well as technological pedagogical content knowledge [64, 65] to their work. Offering mathematics education courses that help teacher candidates develop mathematical habits of mind [31], prepares the candidates to act as mathematicians in their future classrooms. Such action facilitates teaching mathematical problem solving as digital and hands-on (often trial-and-error) experiments. This approach to mathematics teaching and learning, as the paper has demonstrated, goes back to antiquity and spans over the centuries until nowadays when experimentation is supported by digital computation. Recognizing these cultural and historical aspects of mathematics augments teacher education courses with multiple diversity perspectives [66, 67].

The paper provided three major illustrations of different levels of contextual variation and conceptual complexity that are reflection on the author's work as a mathematician-teacher educator. The illustrations were put in context of educational viewpoints suggested over the years by pure mathematicians with deep interest in education [2, 14, 20, 41, 62, 63] as well as by applied mathematicians [6–8, 37]. Those viewpoints have strong emphasis on the power of mathematics as part of education and educational experiment as a mathematical method [51]. Solicited reflections of the author's students aspired to teach mathematics in the schools of North America were included in the paper to demonstrate their appreciation and support of the notion of a mathematics teacher as an applied mathematician.

The paper demonstrated how the use of digital technology, including a spreadsheet, Wolfram Alpha, and the Graphing Calculator, can support teachers' work with students by carrying out a variety of computational experiments. These experiments, designed "to arouse *didactical* understanding" [2, p. 148, italics in the original], included various representations of the farm problem and its artificial modification, the use of the context of accidentally pressing a wrong button on a small calculator by a child as a motivation for exploring possibilities in mathematics [6] leading to comparison of theoretical and experimental probabilities (that is, using experiment to verify theory), recourse to computer graphing as a demonstration of Einstellung effect [61] in solving trigonometric equations, sensitivity of computational knowledge engine Wolfram Alpha to the size of numerical evidence in search for algebraic generalization, and the symmetrical vs. asymmetrical location of roots of Fibonacci-like polynomials. The paper also discussed the use of non-digital technology as support system in the accurate recognition of numerical evidence from hands-on experiments with mathematical ideas.

In conclusion, based on references to multiple sources evincing mathematics teaching as an applied endeavor and drawing on the material of three major illustrations reflecting the author's years of experience using educational technology with teacher candidates, a proposition can be put forward that the education of mathematics teachers to which many mathematicians contributed long before it was advanced to become the field of disciplined inquiry [1], is related to the field of applied mathematics. This relation enables teachers of mathematics not just to cross imaginary "demarcation lines" (ibid, p. 1) of the former field but, rather, through their everyday work, to create many apposite links between the two fields. Whereas mathematics teachers may not necessarily see their work through the lens of applied mathematics, the intent of this paper was to expand the inquiry into mathematics teacher education to allow for that work to be seen through that new lens. It is the author's belief that, due to this vision, the notion of mathematical habit of mind [31] could also be extended to include a similar custom of the applied status.

Conflicts of interest

The author declares no conflict of interest.

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