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RESEARCH ARTICLE

Unified Statistical Framework for Eliminating Parametric Uncertainty in Applied Mathematical Models via Pivotal and Ancillary Quantities

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Abstract: The technique used here emphasizes pivotal quantities and ancillary statistics relevant for obtaining statistical predictive or confidence decisions for anticipated outcomes of applied stochastic models under parametric uncertainty and is applicable whenever the statistical problem is invariant under a group of transformations that acts transitively on the parameter space. It does not require the construction of any tables and is applicable whether the experimental data are complete or Type II censored. The proposed technique is based on a probability transformation and pivotal quantity averaging to solve real-life problems in all areas including engineering, science, industry, automation & robotics, business & finance, medicine and biomedicine. It is conceptually simple and easy to use.

Keywords: anticipated outcomes, parametric uncertainty, unknown (nuisance) parameters, pivotal quantities, ancillary statistics, new-sample prediction, within-sample prediction

1 Introduction

Statistical predictive/confidence decisions (under parametric uncertainty) for future random quantities (e.g., future outcomes, order statistics) based on past and current data are the most prevalent form of statistical inference. Such predictive inferences for future random quantities are widely used in risk management, finance, insurance, economics, hydrology, material sciences, telecommunications, and other industries. Predictive inferences-including predictive distributions, prediction/tolerance limits/intervals, and confidence limits/intervals for future random quantities, based on past and present knowledge-are a fundamental statistical problem, arising in diverse contexts with varied solutions. The approach here is a special case of broader considerations, applicable when the statistical problem is invariant under a transformation group that acts transitively on the parameter space [1–11].

2 Elimination of Parametric Uncertainty

Let $Y = (Y_1 \leq \cdots \leq Y_m)$ be the first m ordered observations (order statistics) in a sample of size h from the two-parameter exponential distribution with the probability density function

$$f_\rho(y) = \vartheta^{-1} \exp\left(-\frac{y-v}{\vartheta}\right), \quad \vartheta > 0,\ v \geq 0, \tag{1}$$

and the cumulative probability distribution function

$$F_\rho(y) = 1 - \exp\left(-\frac{y-v}{\vartheta}\right), \quad \overline{F}_\rho(y) = 1 - F_\rho(y) = \exp\left(-\frac{y-v}{\vartheta}\right), \tag{2}$$

where $\rho = (v, \vartheta)$ is the shift parameter and ϑ is the scale parameter. It is assumed that these parameters are unknown. In Type II censoring, which is of primary interest here, the number of survivors is fixed and Y is a random variable. In this case, the likelihood function is given by

$$\begin{aligned} L(v,\vartheta) &= \prod_{i=1}^{m} f_\rho(y_i)\left(\overline{F}_\rho(y_m)\right)^{h-m} = \frac{1}{\vartheta^m}\exp\left(-\left[\sum_{i=1}^{m}(y_i - v) + (h-m)(y_m - v)\right]/\vartheta\right) \\ &= \frac{1}{\vartheta^m}\exp\left(-\left[\sum_{i=1}^{m}(y_i - y_1 + y_1 - v) + (h-m)(y_m - y_1 + y_1 - v)\right]/\vartheta\right) \\ &= \frac{1}{\vartheta^{m-1}}\exp\left(-\left[\sum_{i=1}^{m}(y_i - y_1) + (h-m)(y_m - y_1)\right]/\vartheta\right) \times \frac{1}{\vartheta}\exp\left(-\frac{h(y_1 - v)}{\vartheta}\right) \\ &= \frac{1}{\vartheta^{m-1}}\exp\left(-\frac{s_m}{\vartheta}\right) \times \frac{1}{\vartheta}\exp\left(-\frac{h(s_1 - v)}{\vartheta}\right), \end{aligned} \tag{3}$$

where

$$\mathbf{S} = \left(S_1 = Y_1,\ S_m = \sum_{i=1}^{m} (Y_i - Y_1) + (h-m)(Y_m - Y_1) \right) \tag{4}$$

is the complete sufficient statistic for ρ. The probability density function of $\mathbf{S} = (S_1, S_m)$ is given by

$$f_\rho(s_1, s_m) = \frac{\frac{1}{\vartheta^{m-1}} \exp\left(-\frac{s_m}{\vartheta}\right) \times \frac{1}{\vartheta} \exp\left(-\frac{h(s_1-\upsilon)}{\vartheta}\right)}{\frac{1}{s_m^{m-2}} \int_0^\infty \frac{s_m^{m-2}}{\vartheta^{m-1}} \exp\left(-\frac{s_m}{\vartheta}\right) ds_m \times \frac{1}{q} \int_0^\infty \frac{h}{\vartheta} \exp\left(-\frac{h(s_1-\upsilon)}{\vartheta}\right) ds_1} = \frac{\frac{1}{\vartheta^{m-1}} \exp\left(-\frac{s_m}{\vartheta}\right) \times \frac{1}{\vartheta} \exp\left(-\frac{h(s_1-\upsilon)}{\vartheta}\right)}{\frac{\Gamma(m-1)}{s_m^{m-2}} \times \frac{1}{h}}$$

$$= \frac{1}{\Gamma(m-1)\vartheta^{m-1}} s_m^{m-2} \exp\left(-\frac{s_m}{\vartheta}\right) \times \frac{h}{\vartheta} \exp\left(-\frac{h(s_1-\upsilon)}{\vartheta}\right) = f_\vartheta(s_m) f_\rho(s_1), \tag{5}$$

where

$$f_\rho(s_1) = \frac{h}{\vartheta} \exp\left(-\frac{h(s_1-\upsilon)}{\vartheta}\right), \quad s_1 \ge \upsilon, \tag{6}$$

$$f_\vartheta(s_m) = \frac{1}{\Gamma(m-1)\vartheta^{m-1}} s_m^{m-2} \exp\left(-\frac{s_m}{\vartheta}\right), \quad s_m \ge 0. \tag{7}$$

$$V_1 = \frac{S_1 - \upsilon}{\vartheta} \tag{8}$$

is the pivotal quantity, the probability density function of which is given by

$$f_1(v_1) = h \exp(-h v_1), \quad v_1 \ge 0, \tag{9}$$

$$V_m = \frac{S_m}{\vartheta} \tag{10}$$

is the pivotal quantity, the probability density function of which is given by

$$f_m(v_m) = \frac{1}{\Gamma(m-1)} v_m^{m-2} \exp(-v_m), \quad v_m \ge 0. \tag{11}$$

3. Adequate Mathematical Models of Cumulative Distribution Functions of Order Statistics for Constructing One-Sided Tolerance Limits (or Two sided Tolerance Interval) in New (Future) Data Samples under Parametric Uncertainty

Theorem 1. Let us assume that $Y_1 \le \ldots \le Y_n$ will be a new (future) random sample of n ordered observations from a known distribution with a probability density function (pdf) $f_\rho(y)$, cumulative distribution function (cdf) $F_\rho(y)$, where ρ is the parameter (in general, vector). Then the adequate mathematical models for a cumulative probability distribution function of the kth order statistic Y_k, $k \in \{1, 2, \ldots, n\}$, to construct one-sided γ − content tolerance limits (or two-sided tolerance interval) for Y_k with confidence level β, are given as follows:

Adequate Applied Mathematical Model 1 of a Cumulative Distribution Function of the kth Order Statistic Y_k is given by

$$\int_0^{F_\rho(y_k)} f_{k,n-k+1}(r)dr = P_\rho(Y_k \le y_k \mid n) = \sum_{j=k}^{n} \binom{n}{j} [F_\rho(y_k)]^j \ [1 - F_\rho(y_k)]^{n-j}. \tag{12}$$

In the above case, a (γ,β) upper, one-sided $\gamma-$ content tolerance limit y_k^U with confidence level β can be obtained by using the following formula:

$$E\left\{\Pr\left(\int_0^{F_\rho(y_k^U)} f_{k,n-k+1}(r)dr \geq \gamma\right)\right\} = E\left\{\Pr\left(P_\rho(Y_k \leq y_k^U \mid n) \geq \gamma\right)\right\} = \beta, \tag{13}$$

where

$$f_{k,n-k+1}(r) = \frac{1}{\mathrm{B}(k,n-k+1)} r^{k-1}(1-r)^{(n-k+1)-1}, \quad 0<r<1, \tag{14}$$

is the probability density function (pdf) of the beta distribution $(Beta(k,n-k+1))$ with the shape parameters k and $n-k+1$.

Proof. It follows from (12) that

$$\frac{d}{dy_k}\int_0^{F_\rho(y_k)} f_{k,n-k+1}(r)dr = \frac{d}{dy_k} P_\rho(Y_k \leq y_k \mid n). \tag{15}$$

This ends the proof.

A (γ,β) lower, one-sided $\gamma-$ content tolerance limit with confidence level β can be obtained by using the following formula:

$$E\left\{\Pr\left(P_\rho(Y_k > y_k^L \mid n) \geq \gamma\right)\right\} = E\left\{\Pr\left(1-\int_0^{F_\mu(y_k^L)} f_{k,n-k+1}(u)du \geq \gamma\right)\right\} = \beta. \tag{16}$$

A (γ,β) two-sided $\gamma-$ content tolerance interval with confidence level β can be obtained by using the following formula:

$$\left[\arg_{y_k^L}\left(E\left\{\Pr\left(P_\rho(Y_k > y_k^L \mid n) \geq \gamma\right)\right\} = \beta\right),\ \arg_{y_k^U}\left(E\left\{\Pr\left(P_\rho(Y_k \leq y_k^U \mid n) \geq \gamma\right)\right\} = \beta\right)\right]$$

$$= \left[\arg_{y_k^L}\left(E\left\{\Pr\left(\int_0^{F_\mu(y_k^L)} f_{k,n-k+1}(r)dr \leq 1-\gamma\right)\right\} = \beta\right),\ \arg_{y_k^U}\left(E\left\{\Pr\left(\int_0^{F_\rho(y_k^U)} f_{k,n-k+1}(r)dr \geq \gamma\right)\right\} = \beta\right)\right]$$

$$= \left[y_k^L,\ y_k^U\right]. \tag{17}$$

Adequate Applied Mathematical Model 2 of a Cumulative Distribution Function of the kth Order Statistic Y_k *is given by*

$$\int_{1-F_\rho(y_k)}^{1} f_{n-k+1,k}(r)dr = P_\rho(Y_k \leq y_k \mid n) = \sum_{j=k}^{n}\binom{n}{j}[F_\rho(y_k)]^j\ [1-F_\rho(y_k)]^{n-j}. \tag{18}$$

In the above case, a (γ,β) upper, one-sided $\gamma-$ content tolerance limit y_k^U with confidence level β can be obtained by using the following formula:

$$E\left\{\Pr\left(\int_{1-F_\rho(y_k^U)}^{1} f_{n-k+1,k}(r)dr \geq \gamma\right) = E\left\{\Pr\left(P_\rho(Y_k \leq y_k^U \mid n) \geq \gamma\right)\right\}\right\} = \beta, \tag{19}$$

where

$$f_{n-k+1,l}(u) = \frac{1}{\mathrm{B}(n-k+1,k)} r^{(n-k+1)-1}(1-r)^{k-1} f_{k,n-k+1}(r), \quad 0<r<1, \tag{20}$$

is the probability density function (pdf) of the beta distribution $(Beta(n-k+1,k))$ with the shape parameters $-$ k+1 and k.

Proof. It follows from (9) that

$$\frac{d}{dy_k}\int\limits_{1-F_\rho(y_k)}^{1} f_{n-k+1,k}(r)dr=\frac{d}{dy_k}P_\rho(Y_k\le y_k\,|\,n). \tag{21}$$

This ends the proof.

A (γ,β) lower, one-sided γ – content tolerance limit with confidence level β can be obtained by using the following formula:

$$E\left\{\Pr\left(P_\rho(Y_k>y_k^L\,|\,n)\ge\gamma\right)\right\}=E\left\{\Pr\left(1-\int\limits_{1-F_\rho(y_k^L)}^{1} f_{n-k+1,k}(r)dr\ge\gamma\right)\right\}=\beta. \tag{22}$$

A (γ,β) two-sided γ – content tolerance interval with confidence level β can be obtained by using the following formula:

$$\left[\arg_{y_k^L}\left(E\left\{\Pr\left(P_\rho(Y_k>y_k^L\,|\,n)\ge\gamma\right)\right\}=\beta\right),\ \arg_{y_k^U}\left(E\left\{\Pr\left(P_\rho(Y_k\le y_k^U\,|\,n)\ge\gamma\right)\right\}=\beta\right)\right]$$

$$=\left[\arg_{y_k^L}\left(E\left\{\Pr\left(\int\limits_{1-F_\rho(y_k^L)}^{1} f_{n-k+1,k}(r)dr\le1-\gamma\right)\right\}=\beta\right),\ \arg_{y_k^U}\left(E\left\{\Pr\left(\int\limits_{1-F_\rho(y_k^U)}^{1} f_{n-k+1,k}(r)dr\ge\gamma\right)\right\}=\beta\right)\right]$$

$$=\left[y_k^L,\ y_k^U\right]. \tag{23}$$

Adequate Applied Mathematical Model 3 of a Cumulative Distribution Function of the kth Order Statistic Y_k *is given by*

$$\int\limits_{0}^{\frac{n-k+1}{k}\frac{F_\rho(y_k)}{1-F_\rho(y_k)}}\varphi_{k,n-k+1}(r)dr=P_\rho(Y_k\le y_k\,|\,n)=\sum_{j=k}^{n}\binom{n}{j}[F_\rho(y_k)]^j\ [1-F_\rho(y_k)]^{n-j}. \tag{24}$$

In the above case, a (γ,β) upper, one-sided γ – content tolerance limit y_k^U with confidence level β can be obtained by using the following formula:

$$E\left\{\Pr\left(\int\limits_{0}^{\frac{n-k+1}{k}\frac{F_\rho(y_k^U)}{1-F_\rho(y_k^U)}}\varphi_{k,n-k+1}(r)dr\ge\gamma\right)\right\}=E\left\{\Pr\left(P_\rho(Y_k\le y_k^U\,|\,n)\ge\gamma\right)\right\}=\beta, \tag{25}$$

where

$$\varphi_{k,n-k+1}(r)=\frac{1}{\mathrm{B}\left(k,n-k+1\right)}\frac{\left[\frac{k}{n-k+1}r\right]^{k-1}}{\left[1+\frac{k}{n-k+1}r\right]^{n+1}}\frac{k}{n-k+1},\quad r\in(0,\infty), \tag{26}$$

is the probability density function (pdf) of the F distribution $(F(k,n-k+1))$ with parameters k and $n-k+1$, which are positive integers known as the degrees of freedom for the numerator and the degrees of freedom for the denominator.

Proof. It follows from (13) that

$$\frac{d}{dy_k}\int\limits_{0}^{\frac{n-k+1}{k}\frac{F_\rho(y_k)}{1-F_\rho(y_k)}}\varphi_{k,n-k+1}(r)dr=\frac{d}{dy_k}P_\rho(Y_k\le y_k\,|\,n). \tag{27}$$

This ends the proof.

A (γ,β) lower, one-sided γ – content tolerance limit with confidence level β can be obtained by using the following formula:

$$E\left\{\Pr\left(P_\rho(Y_k > y_k^L \mid n) \geq \gamma\right)\right\} = E\left\{\Pr\left(1 - \int_0^{\frac{n-k+1}{k}\frac{F_\rho(y_k^L)}{1-F_\rho(y_k^L)}} \varphi_{k,n-k+1}(r)dr \geq \gamma\right)\right\} = \beta. \tag{28}$$

A (γ, β) two-sided $\gamma-$content tolerance interval with confidence level β can be obtained by using the following formula:

$$\left[\arg_{y_k^L}\left(E\left\{\Pr\left(P_\rho(Y_k > y_k^L \mid n) \geq \gamma\right)\right\} = \beta\right),\ \arg_{y_k^U}\left(E\left\{\Pr\left(P_\rho(Y_k \leq y_k^U \mid n) \geq \gamma\right)\right\} = \beta\right)\right]$$

$$= \left[\arg_{y_k^L}\left(E\left\{\Pr\left(\int_0^{\frac{n-k+1}{k}\frac{F_\rho(y_k^L)}{1-F_\rho(y_k^L)}} \varphi_{k,n-k+1}(r)dr \leq 1-\gamma\right)\right\} = \beta\right),\ \arg_{y_k^U}\left(E\left\{\Pr\left(\int_0^{\frac{n-k+1}{k}\frac{F_\rho(y_k^U)}{1-F_\rho(y_k^U)}} \varphi_{k,n-k+1}(r)dr \geq \gamma\right)\right\} = \beta\right)\right]$$

$$= \left[y_k^L,\ y_k^U\right]. \tag{29}$$

Adequate Applied Mathematical Model 4 of a Cumulative Distribution Function of the kth Order Statistic Y_k *is given by*

$$\int_{\frac{k}{n-k+1}\frac{1-F_\rho(y_k)}{F_\rho(y_k)}}^{\infty} \varphi_{n-k+1,k}(r)dr = P_\rho(Y_k \leq y_k \mid n) = \sum_{j=k}^{n}\binom{n}{j}[F_\rho(y_k)]^j\ [1-F_\rho(y_k)]^{n-j}. \tag{30}$$

In the above case, a (γ, β) upper, one-sided $\gamma-$content tolerance limit y_k^U with confidence level β can be obtained by using the following formula:

$$E\left\{\Pr\left(\int_{\frac{k}{n-k+1}\frac{1-F_\rho(y_k^U)}{F_\rho(y_k^U)}}^{\infty} \varphi_{n-k+1,k}(r)dr \geq \gamma\right)\right\} = E\left\{\Pr\left(P_\rho(Y_k \leq y_k^U \mid n) \geq \gamma\right)\right\} = \beta, \tag{31}$$

where

$$\varphi_{n-k+1,k}(r) = \frac{\frac{n-k+1}{k}}{\mathrm{B}(n-k+1,k)} \frac{\left[\frac{n-k+1}{k}r\right]^{n-k}}{\left[1+\frac{n-k+1}{k}r\right]^{n+1}}, \quad r \in (0,\infty), \tag{32}$$

is the probability density function (pdf) of the F distribution $(F(n-k+1,k))$ with parameters $n-k+1$ and k, which are positive integers known as the degrees of freedom for the numerator and the degrees of freedom for the denominator.

Proof. It follows from (30) that

$$\frac{d}{dy_k}\int_{\frac{k}{n-k+1}\frac{1-F_\rho(y_k)}{F_\rho(y_k)}}^{\infty} \varphi_{n-k+1,k}(r)dr = \frac{d}{dy_k}P_\rho(Y_k \leq y_k \mid n). \tag{33}$$

This ends the proof.

A (γ, β) lower, one-sided $\gamma-$content tolerance limit with confidence level β can be obtained by using the following formula:

$$E\left\{\Pr\left(1-\int_{\frac{k}{n-k+1}\frac{1-F_\rho(y_k^L)}{F_\rho(y_k^L)}}^{\infty} \varphi_{n-k+1,k}(r)dr \geq \gamma\right)\right\} = E\left\{\Pr\left(P_\rho(Y_k > y_k^L \mid n) \geq \gamma\right)\right\} = \beta. \tag{34}$$

A (γ,β) two-sided γ – content tolerance interval with confidence level β can be obtained by using the following formula:

$$\left[\arg_{y_k^L}\left(E\left\{\Pr\left(P_\rho(Y_k > y_k^L \mid n)\ge\gamma\right)\right\}=\beta\right),\ \arg_{y_k^U}\left(E\left\{\Pr\left(P_\rho(Y_k \le y_k^U \mid n)\ge\gamma\right)\right\}=\beta\right)\right]$$

$$=\left[\arg_{y_k^L}\left(E\left\{\Pr\left(\int_{\frac{k}{n-k+1}\frac{1-F_\rho(y_k^L)}{F_\rho(y_k^L)}}^{\infty}\varphi_{n-k+1,k}(r)dr\le 1-\gamma\right)\right\}=\beta\right),\ \arg_{y_k^U}\left(E\left\{\Pr\left(\int_{\frac{k}{n-k+1}\frac{1-F_\rho(y_k^U)}{F_\rho(y_k^U)}}^{\infty}\varphi_{n-k+1,k}(r)dr\ge\gamma\right)\right\}=\beta\right)\right]$$

$$=\left[y_k^L,\ y_k^U\right]. \tag{35}$$

4. Adequate Mathematical Models of Conditional Cumulative Distribution Functions of Order Statistic for Constructing One-Sided Tolerance Limits (Or Two-Sided Tolerance Interval) in New (Future) Data Samples under Parametric Uncertainty

Theorem 2. Let us assume that $Y_1\le \ldots \le Y_n$ will be a new (future) random sample of n ordered observations from a known distribution with a probability density function (pdf) $f_\rho(y)$, cumulative distribution function (cdf) $F_\rho(y)$, where ρ is the parameter (in general, vector). Then the adequate mathematical models for a conditional cumulative distribution function (ccdf) of the lth order statistic Y_l, $l\in\{2, \ldots, n\}$, to construct one-sided γ – content tolerance limits (or two-sided tolerance interval) for Y_l $(1 \le k < l \le n)$), given $Y_k=y_k$, with confidence level β, are determined as follows:

Adequate Applied Mathematical Model 5 of a Conditional Cumulative Distribution Function of the lth Order Statistic Y_l is given by

$$\int_0^{1-\frac{\bar F_\rho(y_l)}{\bar F_\rho(y_k)}} f_{l-k,n-l+1}(r)dr = P_\rho(Y_l\le y_l \mid Y_k=y_k;n)=\sum_{j=l-k}^{n-k}\binom{n-k}{j}\left[1-\frac{\bar F_\rho(y_l)}{\bar F_\rho(y_k)}\right]^j\left[\frac{\bar F_\rho(y_l)}{\bar F_\rho(y_k)}\right]^{n-k-j}, \tag{36}$$

In the above case, a (γ,β) upper, one-sided γ – content tolerance limit y_l^U with confidence level β can be obtained by using the following formula:

$$E\left\{\Pr\left(\int_0^{1-\frac{\bar F_\rho(y_l^U)}{\bar F_\rho(y_k)}} f_{l-k,n-l+1}(r)dr\ge\gamma\right)\right\}=E\left\{\Pr\left(P_\rho(Y_l\le y_l^U \mid Y_k=y_k;n)\ge\gamma\right)\right\}=\beta, \tag{37}$$

where $\bar F_\mu(z)=1-F_\mu(z)$,

$$f_{l-k,n-l+1}(r)=\frac{r^{l-k-1}(1-r)^{(n-l+1)-1}}{\mathrm{B}(l-k,n-l+1)},\quad 0<r<1, \tag{38}$$

is the probability density function (pdf) of the beta distribution $(Beta(l-k,n-l+1))$ with shape parameters $l-k$ and $n-l+1$.

Proof. It follows from (36) that

$$\frac{d}{dy_l}\int_0^{1-\frac{\bar F_\rho(y_l)}{\bar F_\rho(y_k)}} f_{l-k,n-l+1}(r)dr=\frac{d}{dy_l}P_\rho(Y_l\le y_l \mid Y_k=y_k;n). \tag{39}$$

This ends the proof.

A (γ,β) lower, one-sided γ – content tolerance limit with confidence level β can be obtained by using the following formula:

$$E\left\{\Pr\left(1-\int_0^{1-\frac{\bar{F}_\rho(y_l^L)}{\bar{F}_\rho(y_k)}} f_{l-k,n-l+1}(r)dr \ge \gamma\right)\right\} = E\left\{\Pr\left(P_\rho(Y_l > y_l^L \mid Y_k = y_k; n) \ge \gamma\right)\right\} = \beta. \quad (40)$$

A (γ,β) two-sided γ – content tolerance interval with confidence level β can be obtained by using the following formula:

$$\left[\arg_{y_l^L}\left(E\left\{\Pr\left(P_\rho(Y_l > y_l^L \mid Y_k = y_k; n) \ge \gamma\right)\right\} = \beta\right),\ \arg_{y_l^U}\left(E\left\{\Pr\left(P_\rho(Y_l \le y_l^U \mid n) \ge \gamma\right)\right\} = \beta\right)\right]$$

$$= \left[\arg_{y_k^L}\left(E\left\{\Pr\left(\int_0^{1-\frac{\bar{F}_\rho(y_l^L)}{\bar{F}_\rho(y_k)}} f_{l-k,n-l+1}(r)dr \le 1-\gamma\right)\right\} = \beta\right),\ \arg_{y_k^U}\left(E\left\{\Pr\left(\int_0^{1-\frac{\bar{F}_\rho(y_l^U)}{\bar{F}_\rho(y_k)}} f_{l-k,n-l+1}(r)dr \ge \gamma\right)\right\} = \beta\right)\right]$$

$$= \left[y_l^L,\ y_l^U\right]. \quad (41)$$

Adequate Applied Mathematical Model 6 of a Conditional Cumulative Distribution Function of the lth Order Statistic Y_l is given by

$$\int_{\frac{\bar{F}_\rho(y_l)}{\bar{F}_\rho(y_k)}}^{1} f_{n-l+1,l-k}(r)dr = P_\rho(Y_l \le y_l \mid Y_k = y_k; n) = \sum_{j=l-k}^{n-k}\binom{n-k}{j}\left[1-\frac{\bar{F}_\rho(y_l)}{\bar{F}_\rho(y_k)}\right]^j\left[\frac{\bar{F}_\rho(y_l)}{\bar{F}_\rho(y_k)}\right]^{n-k-j}, \quad (42)$$

In the above case, a (γ,β) upper, one-sided γ – content tolerance limit y_l^U with confidence level β can be obtained by using the following formula:

$$E\left\{\Pr\left(\int_{\frac{\bar{F}_\rho(y_l^U)}{\bar{F}_\rho(y_k)}}^{1} f_{n-l+1,l-k}(r)dr \ge \gamma\right)\right\} = E\left\{\Pr\left(P_\rho(Y_l \le y_l^U \mid Y_k = y_k; n) \ge \gamma\right)\right\} = \beta, \quad (43)$$

where $\bar{F}_\rho(y) = 1 - F_\rho(y)$,

$$f_{n-l+1,l-k}(r) = \frac{r^{(n-l+1)-1}(1-r)^{l-k-1}}{\mathrm{B}(n-l+1, l-k)}, \quad 0 < r < 1, \quad (44)$$

is the probability density function (pdf) of the beta distribution $(Beta(n-l+1, l-k))$ with shape parameters $n-l+1$ and $l-k$.

Proof. It follows from (42) that

$$\frac{d}{dy_l}\int_{\frac{\bar{F}_\rho(y_l)}{\bar{F}_\rho(y_k)}}^{1} f_{n-l+1,l-k}(r)dr = \frac{d}{dy_l}P_\rho(Y_l \le y_l \mid Y_k = y_k; n). \quad (45)$$

This ends the proof.

A (γ,β) lower, one-sided γ – content tolerance limit with confidence level β can be obtained by using the following formula:

$$E\left\{\Pr\left(1-\int_{\frac{\bar{F}_\rho(y_l^L)}{\bar{F}_\rho(y_k)}}^{1} f_{n-l+1,l-k}(r)dr \ge \gamma\right)\right\} = E\left\{\Pr\left(P_\rho(Y_l > y_l^L \mid Y_k = y_k; n) \ge \gamma\right)\right\} = \beta. \quad (46)$$

A (γ,β) two-sided γ – content tolerance interval with confidence level β can be obtained by using the following formula:

$$\left[\arg_{y_l^L}\left(E\left\{\Pr\left(P_\rho(Y_l > y_l^L \mid Y_k = y_k; n) \geq \gamma\right)\right\} = \beta\right),\ \arg_{y_l^U}\left(E\left\{\Pr\left(P_\rho(Y_l \leq y_l^U \mid Y_k = y_k; n) \geq \gamma\right)\right\} = \beta\right)\right]$$

$$= \left[\arg_{y_l^L}\left(E\left\{\Pr\left(\int_{\frac{\bar{F}_\rho(y_l^L)}{\bar{F}_\rho(y_k)}}^{1} f_{n-l+1,l-k}(r)dr \leq 1-\gamma\right)\right\} = \beta\right),\ \arg_{y_k^U}\left(E\left\{\Pr\left(\int_{\frac{\bar{F}_\rho(y_l^U)}{\bar{F}_\rho(y_k)}}^{1} f_{n-l+1,l-k}(r)dr \geq \gamma\right)\right\} = \beta\right)\right]$$

$$= \left[y_l^L,\ y_l^U\right]. \tag{47}$$

This ends the proof.

Adequate Applied Mathematical Model 7 of a Conditional Cumulative Distribution Function of the lth Order Statistic Y_l is given by

$$\int_0^{\frac{n-l+1}{l-k}\left(1-\frac{\bar{F}_\rho(y_l)}{\bar{F}_\rho(y_k)}\right)\Big/\frac{\bar{F}_\rho(y_l)}{\bar{F}_\rho(y_k)}} f_{l-k,n-l+1}(r)dr = P_\rho(Y_l \leq y_l \mid Y_k = y_k; n)$$

$$= \sum_{j=l-k}^{n-k}\binom{n-k}{j}\left[1-\frac{\bar{F}_\rho(y_l)}{\bar{F}_\rho(y_k)}\right]^j\left[\frac{\bar{F}_\rho(y_l)}{\bar{F}_\rho(y_k)}\right]^{n-k-j}. \tag{48}$$

In the above case, a (γ,β) upper, one-sided γ – content tolerance limit y_l^U with confidence level β can be obtained by using the following formula:

$$E\left\{\Pr\left(\int_0^{\frac{n-l+1}{l-k}\left(1-\frac{\bar{F}_\rho(y_l^U)}{\bar{F}_\rho(y_k)}\right)\Big/\frac{\bar{F}_\rho(y_l^U)}{\bar{F}_\rho(y_k)}} f_{l-k,n-l+1}(r)dr \geq \gamma\right)\right\} = E\left\{\Pr\left(P_\rho(Y_l \leq y_l^U \mid Y_k = y_k; n) \geq \gamma\right)\right\} = \beta, \tag{49}$$

where $\bar{F}_\rho(y) = 1 - F_\rho(y)$,

$$f_{l-k,n-l+1}(r) = \frac{\frac{l-k}{n-l+1}}{\mathrm{B}(l-k, n-l+1)}\frac{\left[\frac{l-k}{n-l+1}r\right]^{l-k-1}}{\left[1+\frac{l-k}{n-l+1}r\right]^{n-k+1}}, \quad r \in (0,\infty), \tag{50}$$

is the probability density function (pdf) of the F distribution $(F(l-k, n-l+1))$ with parameters $l-k$ and $n-l+1$, which are positive integers known as the degrees of freedom for the numerator and the degrees of freedom for the denominator.

Proof. It follows from (48) that

$$\frac{d}{dy_l}\int_0^{\frac{n-l+1}{l-k}\left(1-\frac{\bar{F}_\rho(y_l)}{\bar{F}_\rho(y_k)}\right)\Big/\frac{\bar{F}_\rho(y_l)}{\bar{F}_\rho(y_k)}} f_{l-k,n-l+1}(r)dr = \frac{d}{dy_l}P_\rho(Y_l \leq y_l \mid Y_k = y_k; n). \tag{51}$$

This ends the proof.

A (γ,β) lower, one-sided γ – content tolerance limit with confidence level β can be obtained by using the following formula:

$$E\left\{\Pr\left(1-\int_0^{\frac{n-l+1}{l-k}\left(1-\frac{\bar{F}_\rho(y_l^L)}{\bar{F}_\rho(y_k)}\right)\Big/\frac{\bar{F}_\rho(y_l^L)}{\bar{F}_\rho(y_k)}} f_{l-k,n-l+1}(r)dr \geq \gamma\right)\right\} = E\left\{\Pr\left(P_\rho(Y_l > y_l^L \mid Y_k = y_k; n) \geq \gamma\right)\right\} = \beta. \tag{52}$$

A (γ, β) two-sided γ – content tolerance interval with confidence level β can be obtained by using the following formula:

$$\left[\arg_{y_l^L}\left(E\left\{\Pr\left(P_\rho(Y_l > y_l^L \mid Y_k = y_k; n) \geq \gamma\right)\right\} = \beta\right),\ \arg_{y_l^U}\left(E\left\{\Pr\left(\Pr\left(P_\rho(Y_l \leq y_l^U \mid Y_k = y_k; n) \geq \gamma\right)\right)\right\} = \beta\right)\right]$$

$$= \left[\begin{array}{l} \arg_{y_l^L}\left(E\left\{\Pr\left(\int_0^{\frac{n-l+1}{l-k}\left(1-\frac{\bar{F}_\rho(y_l^L)}{\bar{F}_\rho(y_k)}\right)\Big/\frac{\bar{F}_\rho(y_l^L)}{\bar{F}_\rho(y_k)}} f_{l-k,n-l+1}(r)dr \leq 1-\gamma\right)\right\} = \beta\right), \\ \arg_{y_k^U}\left(E\left\{\Pr\left(\int_0^{\frac{n-l+1}{l-k}\left(1-\frac{\bar{F}_\rho(y_l^U)}{\bar{F}_\rho(y_k)}\right)\Big/\frac{\bar{F}_\rho(y_l^U)}{\bar{F}_\rho(y_k)}} f_{l-k,n-l+1}(r)dr \geq \gamma\right)\right\} = \beta\right) \end{array}\right] = \left[y_l^L, y_l^U\right]. \tag{53}$$

This ends the proof.

Adequate Applied Mathematical Model 8 of a Conditional Cumulative Distribution Function of the lth Order Statistic Y_l is given by

$$\int_{\frac{l-k}{n-l+1}\frac{\bar{F}_\rho(y_l)}{\bar{F}_\rho(y_k)}\Big/\left(1-\frac{\bar{F}_\rho(y_l)}{\bar{F}_\rho(y_k)}\right)}^{\infty} f_{n-l+1,l-k,}(r)dr = P_\rho(Y_l \leq y_l \mid Y_k = y_k; n)$$

$$= \sum_{j=l-k}^{n-k}\binom{n-k}{j}\left[1-\frac{\bar{F}_\rho(y_l)}{\bar{F}_\rho(y_k)}\right]^j\left[\frac{\bar{F}_\rho(y_l)}{\bar{F}_\rho(z_k)}\right]^{n-k-j} \tag{54}$$

In the above case, a (γ, β) upper, one-sided γ – content tolerance limit y_l^U with confidence level β can be obtained by using the following formula:

$$E\left\{\Pr\left(\int_{\frac{l-k}{n-l+1}\frac{\bar{F}_\rho(y_l^U)}{\bar{F}_\rho(y_k)}\Big/\left(1-\frac{\bar{F}_\rho(y_l^U)}{\bar{F}_\rho(y_k)}\right)}^{\infty} f_{n-l+1,l-k,}(r)dr \geq \gamma\right)\right\} = E\left\{\Pr\left(P_\rho(Y_l \leq y_l^U \mid Y_k = y_k; n) \geq \gamma\right)\right\} = \beta, \tag{55}$$

where $\bar{F}_\rho(y) = 1 - F_\rho(y)$,

$$f_{n-l+1,l-k}(r) = \frac{\dfrac{l-k}{n-l+1}}{\mathrm{B}(l-k, n-l+1)}\frac{\left[\dfrac{l-k}{n-l+1}r\right]^{l-k-1}}{\left[1+\dfrac{l-k}{n-l+1}r\right]^{n-k+1}}, \quad r \in (0, \infty), \tag{56}$$

is the probability density function (pdf) of the F distribution $(F(n-l+1, l-k))$ with parameters $n - l+1$ and $l - k$, which are positive integers known as the degrees of freedom for the numerator and the degrees of freedom for the denominator.

Proof. It follows from (54) that

$$\frac{d}{dy_l}\int\limits_{\frac{l-k}{n-l+1}\frac{\bar{F}_\rho(y_l)}{\bar{F}_\rho(y_k)}\Big/\left(1-\frac{\bar{F}_\rho(y_l)}{\bar{F}_\rho(y_k)}\right)}^{\infty} f_{n-l+1,l-k,}(r)dr = \frac{d}{dy_l}P_\rho(Y_l \le y_l \,|\, Y_k = y_k; n). \tag{57}$$

This ends the proof.

A (γ,β) lower, one-sided γ – content tolerance limit with confidence level β can be obtained by using the following formula:

$$E\left\{\Pr\left(1-\int\limits_{\frac{l-k}{n-l+1}\frac{\bar{F}_\rho(y_l^L)}{\bar{F}_\rho(y_k)}\Big/\left(1-\frac{\bar{F}_\rho(y_l^L)}{\bar{F}_\rho(y_k)}\right)}^{\infty} f_{n-l+1,l-k,}(r)dr \ge \gamma\right)\right\} = E\left\{\Pr\left(P_\rho(Y_l > y_l^L \,|\, Y_k = y_k; n) \ge \gamma\right)\right\} = \beta. \tag{58}$$

A (γ,β) two-sided γ – content tolerance interval with confidence level β can be obtained by using the following formula:

$$\left[\arg_{y_l^L}\left(E\left\{\Pr\left(P_\rho(Y_l > y_l^L \,|\, Y_k = y_k; n) \ge \gamma\right)\right\} = \beta\right),\ \arg_{y_l^U}\left(E\left\{\Pr\left(\Pr\left(P_\rho(Y_l \le y_l^U \,|\, Y_k = y_k; n) \ge \gamma\right)\right)\right\} = \beta\right)\right]$$

$$= \left[\begin{array}{l} \arg_{y_l^L}\left(E\left\{\Pr\left(\int\limits_{\frac{l-k}{n-l+1}\frac{\bar{F}_\rho(y_l^L)}{\bar{F}_\rho(y_k)}\Big/\left(1-\frac{\bar{F}_\rho(y_l^L)}{\bar{F}_\rho(y_k)}\right)}^{\infty} f_{n-l+1,l-k,}(r)dr \le 1-\gamma\right)\right\} = \beta\right), \\ \arg_{y_k^U}\left(E\left\{\Pr\left(\int\limits_{\frac{l-k}{n-l+1}\frac{\bar{F}_\rho(y_l^U)}{\bar{F}_\rho(y_k)}\Big/\left(1-\frac{\bar{F}_\rho(y_l^U)}{\bar{F}_\rho(y_k)}\right)}^{\infty} f_{n-l+1,l-k,}(r)dr \ge \gamma\right)\right\} = \beta\right) \end{array}\right] = \left[y_l^L,\ y_l^U\right]. \tag{59}$$

This ends the proof.

5. Two-Parameter Exponential Distribution

Let $\mathbf{Y} = (Y_1 \le \ldots \le Y_m)$ be the first m ordered observations (order statistics) in a sample of size h from the two-parameter exponential distribution with the probability density function

$$f_\rho(y) = \vartheta^{-1}\exp\left(-\frac{y-\upsilon}{\vartheta}\right), \quad \vartheta > 0,\ \upsilon \ge 0, \tag{60}$$

and the cumulative probability distribution function

$$F_\rho(y) = 1-\exp\left(-\frac{y-\upsilon}{\vartheta}\right), \quad \bar{F}_\rho(y) = 1-F_\rho(y) = \exp\left(-\frac{y-\upsilon}{\vartheta}\right), \tag{61}$$

where $\rho = (\upsilon, \vartheta)$, υ is the shift parameter and ϑ is the scale parameter. It is assumed that these parameters are unknown. In Type II censoring, which is of primary interest here, the number of survivors is fixed and Y is a random variable. In this case, the likelihood function is given by

$$L(\upsilon,\vartheta) = \prod_{i=1}^{m} f_\rho(y_i)\left(\bar{F}_\rho(y_m)\right)^{h-m} = \frac{1}{\vartheta^m}\exp\left(-\left[\sum_{i=1}^{m}(y_i-\upsilon)+(h-m)(y_m-\upsilon)\right]\Big/\vartheta\right)$$

$$= \frac{1}{\vartheta^m}\exp\left(-\left[\sum_{i=1}^{m}(y_i-y_1+y_1-\upsilon)+(h-m)(y_m-y_1+y_1-\upsilon)\right]\Big/\vartheta\right)$$

$$=\frac{1}{\vartheta^{m-1}}\exp\left(-\left[\sum_{i=1}^{m}(y_i-y_1)+(h-m)(y_m-y_1)\right]\Big/\vartheta\right)$$

$$\times\frac{1}{\vartheta}\exp\left(-\frac{h(y_1-\upsilon)}{\vartheta}\right)=\frac{1}{\vartheta^{m-1}}\exp\left(-\frac{s_m}{\vartheta}\right)\times\frac{1}{\vartheta}\exp\left(-\frac{h(s_1-\upsilon)}{\vartheta}\right), \tag{62}$$

where

$$\mathbf{S}=\left(S_1=Y_1,\ S_m=\sum_{i=1}^{m}(Y_i-Y_1)+(h-m)(Y_m-Y_1)\right) \tag{63}$$

is the complete sufficient statistic for ρ. The probability density function of $\mathbf{S}=(S_1, S_m)$ is given by

$$f_\rho(s_1,s_m)=\frac{\frac{1}{\vartheta^{m-1}}\exp\left(-\frac{s_m}{\vartheta}\right)\times\frac{1}{\vartheta}\exp\left(-\frac{h(s_1-\upsilon)}{\vartheta}\right)}{\frac{1}{s_m^{m-2}}\int_0^\infty\frac{s_m^{m-2}}{\vartheta^{m-1}}\exp\left(-\frac{s_m}{\vartheta}\right)ds_m\times\frac{1}{q}\int_0^\infty\frac{h}{\vartheta}\exp\left(-\frac{h(s_1-\upsilon)}{\vartheta}\right)ds_1}$$

$$=\frac{\frac{1}{\vartheta^{m-1}}\exp\left(-\frac{s_m}{\vartheta}\right)\times\frac{1}{\vartheta}\exp\left(-\frac{h(s_1-\upsilon)}{\vartheta}\right)}{\frac{\Gamma(m-1)}{s_m^{m-2}}\times\frac{1}{h}}$$

$$=\frac{1}{\Gamma(m-1)\vartheta^{m-1}}s_m^{m-2}\exp\left(-\frac{s_m}{\vartheta}\right)\times\frac{h}{\vartheta}\exp\left(-\frac{h(s_1-\upsilon)}{\vartheta}\right)=f_\vartheta(s_m)\,f_\rho(s_1), \tag{64}$$

where

$$f_\rho(s_1)=\frac{h}{\vartheta}\exp\left(-\frac{h(s_1-\upsilon)}{\vartheta}\right),\quad s_1\ge\upsilon, \tag{65}$$

$$f_\vartheta(s_m)=\frac{1}{\Gamma(m-1)\vartheta^{m-1}}s_m^{m-2}\exp\left(-\frac{s_m}{\vartheta}\right),\quad s_m\ge 0. \tag{66}$$

$$V_1=\frac{S_1-\upsilon}{\vartheta} \tag{67}$$

is the pivotal quantity, the probability density function of which is given by

$$f_1(v_1)=h\exp(-hv_1),\quad v_1\ge 0, \tag{68}$$

$$V_m=\frac{S_m}{\vartheta} \tag{69}$$

is the pivotal quantity, the probability density function of which is given by

$$f_m(v_m)=\frac{1}{\Gamma(m-1)}v_m^{m-2}\exp(-v_m),\quad v_m\ge 0. \tag{70}$$

Constructing a (γ,β) upper, one-sided γ – content tolerance limit with confidence level β for the case of Model 1

Theorem 3. Let $Y_1\le\ldots\le Y_m$ be the first m ordered observations from the preliminary sample of size h from a two-parameter exponential distribution defined by the probability density function (49). Then the upper one-sided γ-content tolerance limit (with a confidence level β) y_k^U on the kth order statistic Y_k from a set of n future ordered observations $Y_1\le\ldots\le Y_n$ also from the distribution (49), which satisfies

$$E\left\{\Pr\left(P_\rho(Y_k\le y_k^U\mid n)\ge\gamma\right)\right\}=\beta, \tag{71}$$

is given by

$$y_k^U = \begin{cases} S_1 + \dfrac{S_m}{h}\left[1-\left(\dfrac{\Omega_\gamma^h}{\beta}\right)^{\frac{1}{m-1}}\right], & \text{if } \left(\dfrac{\Omega_\gamma^h}{\beta}\right)^{\frac{1}{m-1}} \le 1, \\ S_1 + \dfrac{S_m}{h}\left[\left(\dfrac{\Omega_\gamma^h}{\beta}\right)^{\frac{1}{m-1}} - 1\right], & \text{if } \left(\dfrac{\Omega_\gamma^h}{\beta}\right)^{\frac{1}{m-1}} > 1, \end{cases} \tag{72}$$

where

$$\Omega_\gamma = 1 - q_{(k,n-k+1),\gamma}\left(Beta(k, n\text{-}k+1),\ \gamma \text{ quantile}\right). \tag{73}$$

Proof. It follows from (71), (72) and (73) that

$$E\left\{\Pr\left(P_\rho(Y_k \le y_k^U \mid n) \ge \gamma\right)\right\}$$

$$= E\left\{\Pr\left(\int_0^{F_\rho(y_k^U)} f_{k,n-k+1}(r)dr \ge \gamma\right)\right\} = E\left\{\Pr\left(1-\exp\left(-\frac{y_k^U - \upsilon}{\vartheta}\right) \ge q_{k,n-k+1;\gamma}\right)\right\}$$

$$= E\left\{\Pr\left(\exp\left(-\frac{y_k^U - \upsilon}{\vartheta}\right) \le 1 - q_{k,n-k+1;\gamma}\right)\right\}$$

$$= E\left\{\Pr\left(-\frac{y_k^U - \upsilon}{\vartheta} \le \ln\left(1-q_{k,n-k+1;\gamma}\right)\right)\right\} = E\left\{\Pr\left(\frac{y_k^U - \upsilon}{\vartheta} \ge -\ln\left(1-q_{k,n-k+1;\gamma}\right)\right)\right\}$$

$$= E\left\{\Pr\left(\frac{y_k^U - S_1}{S_m}\frac{S_m}{\vartheta} + \frac{S_1 - \upsilon}{\vartheta} \ge -\ln\left(1-q_{k,n-k+1;\gamma}\right)\right)\right\}$$

$$= E\left\{\Pr\left(\frac{S_1 - \upsilon}{\vartheta} \ge -\frac{y_k^U - S_1}{S_m}\frac{S_m}{\vartheta} - \ln\left(1-q_{k,n-k+1;\gamma}\right)\right)\right\}$$

$$= E\left\{\Pr\left(V_1 \ge -\eta_k^U V_m - \ln\Omega_\gamma\right)\right\} = E\left\{1 - \Pr\left(V_1 \le -\eta_k^U V_m - \ln\Omega_\gamma\right)\right\} = E\left\{1 - \int_0^{-\eta_k^U V_m - \ln\Omega_\gamma} f_1(v_1)dv_1\right\}, \tag{74}$$

where

$$\eta_k^U = \frac{y_k^U - S_1}{S_m}. \tag{75}$$

It follows from (74) and (75) that

$$E\left\{1 - \int_0^{-\eta_k^U V_m - \ln\Omega_\gamma} f_1(v_1)dv_1\right\} = E\left\{1 - \int_0^{-\eta_k^U V_m - \ln\Omega_\gamma} h\exp(-hv_1)dv_1\right\}$$

$$= E\left\{1 - \left[1 - \exp\left(-h\left[-\eta_k^U V_m - \ln\Omega_\gamma\right]\right)\right]\right\} = E\left\{\exp\left(h\eta_k^U V_m\right)\exp\left(\ln\Omega_\gamma^h\right)\right\} = E\left\{\Omega_\gamma^h \exp\left(h\eta_k^U V_m\right)\right\}$$

$$= \int_0^\infty \left(\Omega_\gamma^h \exp\left(h\eta_k^U v_m\right)\right) f_m(v_m)dv_m$$

$$= \int_0^\infty \left(\Omega_\gamma^h \exp\left(h\eta_k^U v_m\right)\right)\frac{1}{\Gamma(m-1)} v_m^{m-2}\exp\left(-v_m\right)dv_m = \Omega_\gamma^h \int_0^\infty \frac{1}{\Gamma(m-1)} v_m^{m-2}\exp\left(-v_m\left[1-h\eta_k^U\right]\right)dv_m$$

$$= \frac{\Omega_\gamma^h}{\left[1-h\eta_k^U\right]^{m-1}} = \beta. \tag{76}$$

It follows from (75) and (76) that

$$\eta_k^U = \frac{y_k^U - S_1}{S_m} = \frac{1}{h}\left(1 - \left[\frac{\Omega_\gamma^h}{\beta}\right]^{\frac{1}{m-1}}\right). \tag{77}$$

It follows from (77) that

$$y_k^U = S_1 + \frac{S_m}{h}\left(1 - \left[\frac{\Omega_\gamma^h}{\beta}\right]^{\frac{1}{m-1}}\right). \tag{78}$$

Then (72) follows from (78). This ends the proof.

Constructing a (γ, β) lower, one-sided γ – content tolerance limit with confidence level β for the case of Model 1

Theorem 4. Let $Y_1 \le \ldots \le Y_m$ be the first m ordered observations from the preliminary sample of size h from a two-parameter exponential distribution defined by the probability density function (60). Then the lower one-sided γ-content tolerance limit (with a confidence level β) y_k^L on the kth order statistic Y_k from a set of n future ordered observations $Y_1 \le \ldots \le Y_n$ also from the distribution (60)), which satisfies

$$E\left\{\Pr\left(P_\mu(Y_k > y_k^L \mid n) \ge \gamma\right)\right\} = \beta, \tag{79}$$

is given by

$$y_k^L = \begin{cases} S_1 + \dfrac{S_m}{h}\left[1 - \left(\dfrac{\Omega_{1-\gamma}^h}{1-\beta}\right)^{\frac{1}{m-1}}\right], & \text{if } \left(\dfrac{\Omega_{1-\gamma}^h}{1-\beta}\right)^{\frac{1}{m-1}} \le 1, \\ S_1 + \dfrac{S_m}{h}\left[\left(\dfrac{\Omega_{1-\gamma}^h}{1-\beta}\right)^{\frac{1}{m-1}} - 1\right], & \text{if } \left(\dfrac{\Omega_{1-\gamma}^h}{1-\beta}\right)^{\frac{1}{m-1}} > 1, \end{cases} \tag{80}$$

where

$$\Omega_{1-\gamma} = 1 - q_{(k,n-k+1),1-\gamma}\left(Beta(k, n\text{-}k+1),\ 1-\gamma \ \text{ quantile}\right). \tag{81}$$

Proof. It follows from (79) and (81) that

$$E\left\{\Pr\left(P_\rho(Y_k > y_k^L \mid n) \ge \gamma\right)\right\} = E\left\{\Pr\left(\int_0^{F_\rho(y_k^L)} f_{k,n-k+1}(r)dr \le 1-\gamma\right)\right\}$$

$$= E\left\{\Pr\left(\exp\left(-\frac{y_k^L - \upsilon}{\vartheta}\right) \ge 1 - q_{k,n-k+1;1-\gamma}\right)\right\}$$

$$= E\left\{\Pr\left(\frac{y_k^L - S_1}{S_m}\frac{S_m}{\vartheta} + \frac{S_1 - \upsilon}{\vartheta} \le -\ln\left(1 - q_{k,n-k+1;1-\gamma}\right)\right)\right\}$$

$$= E\left\{\Pr\left(\frac{S_1 - \upsilon}{\vartheta} \le -\frac{y_k^L - S_1}{S_m}\frac{S_m}{\vartheta} - \ln\left(1 - q_{k,n-k+1;1-\gamma}\right)\right)\right\}$$

$$= E\left\{\Pr\left(V_1 \le -\eta_k^L V_m - \ln\Omega_{1-\gamma}\right)\right\} = E\left\{\int_0^{-\eta_k^L V_m - \ln\Omega_{1-\gamma}} f_1(v_1)dv_1\right\}, \tag{82}$$

where

$$\eta_k^L = \frac{y_k^L - S_1}{S_m}. \tag{83}$$

It follows from (68) and (82) that

$$E\left\{\int_0^{-\eta_k^L V_m - \ln\Omega_{1-\gamma}} f_1(v_1)dv_1\right\} = E\left\{\int_0^{-\eta_k^L V_m - \ln\Omega_{1-\gamma}} h\exp(-hv_1)dv_1\right\}$$

$$= E\left\{1-\exp\left(-h\left[-\eta_k^L V_m - \ln\Omega_{1-\gamma}\right]\right)\right\} = E\left\{1-\exp\left(h\eta_k^L V_m\right)\exp\left(q\ln\Omega_{1-\gamma}\right)\right\} = E\left\{1-\Omega_{1-\gamma}^h \exp\left(h\eta_k^L V_m\right)\right\}$$

$$= \int_0^\infty \left(1-\Omega_{1-\gamma}^h \exp\left(h\eta_k^L v_m\right)\right) f_m(v_m)dv_m$$

$$= \int_0^\infty \left(1-\Omega_{1-\gamma}^h \exp\left(h\eta_k^L v_m\right)\right)\frac{1}{\Gamma(m-1)} v_m^{m-2}\exp(-v_m)dv_m = 1-\Omega_{1-\gamma}^h \int_0^\infty \frac{1}{\Gamma(m-1)} v_m^{m-2}\exp\left(-v_m\left[1-h\eta_k^L\right]\right)dv_m$$

$$= 1-\frac{\Omega_{1-\gamma}^h}{\left[1-h\eta_k^L\right]^{m-1}} = \beta. \tag{84}$$

It follows from (83) and (84) that

$$\eta_{L_k} = \frac{y_k^L - S_1}{S_m} = \frac{1}{h}\left(1-\left[\frac{\Omega_{1-\gamma}^h}{1-\beta}\right]^{\frac{1}{m-1}}\right). \tag{85}$$

It follows from (85) that

$$y_k^L = S_1 + \frac{S_m}{h}\left(1-\left[\frac{\Omega_{1-\gamma}^h}{1-\beta}\right]^{\frac{1}{m-1}}\right). \tag{86}$$

Then (80) follows from (86). This ends the proof.

A. *Numerical Practical Example*

Let us assume that k =5, m =8, h =10, n=12, $\gamma = \beta$ = 0.95,

$$\mathbf{S} = \left(S_1 = Y_1 = 9,\ S_m = \sum_{i=1}^m (Y_i - Y_1) + (h-m)(Y_m - Y_1)\right)$$
$$= \left(S_1 = 9,\ S_m = 0+1+2+4+6+10+15+23+(10-8)23 = 107\right), \tag{87}$$

Then , the $(\gamma = 0.95, \beta = 0.95)$ upper, one-sided γ – content tolerance limit y_k^U with confidence level β can be obtained from (72), where the quantile of $Beta(k, n\text{-}k+1), \gamma$ is given by

$$q_{(k,n-k+1),\gamma} = 0.609138, \tag{88}$$

$$\Omega_{1-\gamma} = 1 - q_{(k,n-k+1),1-\gamma} = 1-0.609138 = 0.390862. \tag{89}$$

It follows from (72), (87) and (89) that

$$y_k^U = S_1 + \frac{S_m}{h}\left[1-\left(\frac{\Omega_\gamma^h}{\beta}\right)^{\frac{1}{m-1}}\right] = 9 + \frac{107}{10}\left[\left(1-\frac{[0.390862]^{10}}{0.95}\right)^{\frac{1}{8-1}}\right] = 9 + 7.883285 = 16.883285. \tag{90}$$

The $(\gamma = 0.95, \beta = 0.95)$ lower, one-sided γ – content tolerance limit y_k^L with confidence level β can be obtained from (80), where the quantile of $Beta(k, n\text{-}k+1), 1-\gamma$ is given by

$$q_{(k,n-k+1),1-\gamma} = 0.181025, \tag{91}$$

$$\Omega_{1-\gamma} = 1 - q_{(k,n-k+1),1-\gamma} = 1-0.181025 = 0.818975. \tag{92}$$

It follows from (80), (87) and (92) that

$$y_k^L = S_1 + \frac{S_m}{h}\left[\left(\frac{\Omega_\gamma^h}{1-\beta}\right)^{\frac{1}{m-1}} - 1\right] = 9 + \frac{107}{10}\left[\left(\frac{[0.818975]^{10}}{1-0.95}\right)^{\frac{1}{8-1}} - 1\right]$$

$$= 9 + \frac{107}{10}[1.15335326 - 1] = 10.64088. \quad (93)$$

The $(\gamma = 0.95, \beta = 0.95)$ two-sided γ – content tolerance interval with confidence level β can be obtained by using (90) and (93):

$$\left[y_k^L,\, y_k^U\right] = [10.64088,\ 16.883285]. \quad (94)$$

6. New Intelligent Transformation Technique for Derivation of the Density Function of the Student's T Distribution

Theorem 5. If $W_1 \in N(0,1)$ and $W_2 \in \chi^2(\upsilon)$ are independent random variables, then

$$W_1 / \sqrt{W_2/\upsilon} = T(\upsilon), \quad (95)$$

where $t(\upsilon)$ follows the student's *t* distribution with υ degrees of freedom,

$$t(\upsilon) \sim f(t) = \frac{\Gamma\left((\upsilon+1)/2)\right)}{\sqrt{\pi\upsilon}\,\Gamma(\upsilon/2)}\left[1+\frac{t^2}{\upsilon}\right]^{-(\upsilon+1)/2}, \quad -\infty < t < \infty. \quad (96)$$

Proof.

$$w_1 \sim f_1(w_1) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{w_1^2}{2}\right), \quad -\infty < w_1 < \infty, \quad (97)$$

where

$$w_1 = t\left[\frac{w_2}{\upsilon}\right]^{1/2}, \quad dw_1 = \left[\frac{w_2}{\upsilon}\right]^{1/2} dt. \quad (98)$$

It follows from (97) and (98) that

$$f_1(w_1)dw_1 = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{w_1^2}{2}\right)dw_1 = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{t^2[w_2/\upsilon]}{2}\right)\left[\frac{w_2}{\upsilon}\right]^{1/2} dt = f(t \mid w_2)dt, -\infty < t < \infty. \quad (99)$$

$$w_2 \sim f_2(w_2) = \frac{1}{\Gamma(\upsilon/2)2^{\upsilon/2}} w_2^{(\upsilon/2)-1}\exp\left(-\frac{w_2}{2}\right), \quad 0 < w_2 < \infty. \quad (100)$$

It follows from (99) and (100) that

$$f(t) = \int_0^\infty f(t \mid w_2)\, f_2(w_2)dw_2$$

$$= \int_0^\infty \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{t^2[w_2/\upsilon]}{2}\right)\left[\frac{w_2}{\upsilon}\right]^{1/2}\frac{1}{\Gamma(\upsilon/2)2^{\upsilon/2}} w_2^{(\upsilon/2)-1}\exp\left(-\frac{w_2}{2}\right)dw_2$$

$$= \int_0^\infty \frac{1}{\sqrt{\pi\upsilon}\Gamma(\upsilon/2)2^{(\upsilon+1)/2}} w_2^{((\upsilon+1)/2)-1}\exp\left(-\frac{w_2}{2}\left[1+\frac{t^2}{\upsilon}\right]\right)dw_2 = \frac{\Gamma\left((\upsilon+1)/2)\right)}{\sqrt{\pi\upsilon}\,\Gamma(\upsilon/2)}\left[1+\frac{t^2}{\upsilon}\right]^{-(\upsilon+1)/2}, -\infty < t < \infty. \quad (101)$$

This ends the proof.

7. Confidence Interval for the Difference of Means of Two Different Normal Populations

In most applications, two populations are compared using the difference in the means. Let U_1, U_2, ..., U_m be a sample of size m from a normal population having mean μ_m and variance σ_m^2 and let Z_1, ..., Z_n be a sample of size n from a different normal population having mean μ_n and variance σ_n^2 and suppose that the two samples are independent of each other. We are interested in constructing a confidence interval for $\mu_m - \mu_n$. To obtain this confidence interval, we need the distribution of $\bar{U}_m - \bar{Z}_n$, where

$$\bar{U}_m = \sum_{i=1}^{m} U_i \Big/ m \sim N\left(\mu_m, \sigma_m^2/m\right), \quad \bar{Z}_n = \sum_{i=1}^{m} Z_i \Big/ n \sim N\left(\mu_n, \sigma_n^2/n\right). \tag{102}$$

It follows from (102) that

$$\bar{U}_m - \bar{Z}_n \sim N\left(\mu_m - \mu_n, \frac{\sigma_m^2}{m} + \frac{\sigma_n^2}{n}\right). \tag{103}$$

It follows from (103) that

$$\frac{\bar{U}_m - \bar{Z}_n - (\mu_m - \mu_n)}{\sigma_m^2 / m + \sigma_n^2 / n} = W_1 \sim N(0,1). \tag{104}$$

This is independent of

$$\sum_{i=1}^{m} \left(U_i - \bar{U}_m\right)^2 \Big/ \sigma_m^2 = \frac{(m-1)}{\sigma_m^2} \frac{\sum_{i=1}^{m} \left(U_i - \bar{U}_m\right)^2}{(m-1)} = \frac{(m-1)S_m^2}{\sigma_m^2} \sim \chi_{m-1}^2 \tag{105}$$

and

$$\sum_{i=1}^{n} \left(Z_i - \bar{Z}_n\right)^2 \Big/ \sigma_n^2 = \frac{(n-1)}{\sigma_n^2} \frac{\sum_{i=1}^{n} \left(Z_i - Z_n\right)^2}{(n-1)} = \frac{(n-1)S_n^2}{\sigma_n^2} \sim \chi_{n-1}^2, \tag{106}$$

where

$$\frac{(m-1)S_m^2}{\sigma_m^2} + \frac{(n-1)S_n^2}{\sigma_n^2} = W_2 \sim \chi^2(m+n-2). \tag{107}$$

Taking (95), (104) and (107) into account, we have that

$$\frac{W_1}{\sqrt{W_2 / (m+n-2)}} = \frac{\dfrac{\bar{U}_m - \bar{Z}_n - (\mu_m - \mu_n)}{\sigma_m^2 / m + \sigma_n^2 / n}}{\sqrt{\left[\dfrac{(m-1)S_m^2}{\sigma_m^2} + \dfrac{(n-1)S_n^2}{\sigma_n^2}\right] \Big/ (m+n-2)}}$$

$$= \frac{\bar{U}_m - \bar{Z}_n - (\mu_m - \mu_n)}{\sqrt{(m-1)S_m^2 / \sigma_m^2 + (n-1)S_n^2 / \sigma_n^2}} \sqrt{\frac{m+n-2}{\sigma_m^2 / m + \sigma_n^2 / n}} = T(m+n-2) \sim f(t), \tag{108}$$

where $T(m+n-2)$ is a t-random variable with $m + n – 2$ degrees of freedom,

$$f(t) = \frac{\Gamma\left((m+n-1)/2\right)}{\sqrt{\pi(m+n-2)}\ \Gamma\left((m+n-2)/2\right)} \left[1 + \frac{t^2}{m+n-2}\right]^{-(m+n-1)/2}, \quad -\infty < t < \infty. \tag{109}$$

Using (108) and (109), it can be obtained a 100(1−α)% confidence interval for $\bar{U}_m - \bar{Z}_n - (\mu_m - \mu_n)$ from

$$P\left(t_1 \le T(m+n-2) \le t_2\right) = P\left(t_1 \le \frac{\bar{U}_m - \bar{Z}_n - (\mu_m - \mu_n)}{\sqrt{(m-1)S_m^2 / \sigma_m^2 + (n-1)S_n^2 / \sigma_n^2}} \frac{\sqrt{m+n-2}}{\sqrt{\sigma_m^2 / m + \sigma_n^2 / n}} \le t_2\right)$$

$$= P\left(t_1\sqrt{\frac{(m-1)S_m^2/\sigma_m^2+(n-1)S_n^2/\sigma_n^2}{m+n-2}}\sqrt{\sigma_m^2/m+\sigma_n^2/n} \le \bar{U}_m-\bar{Z}_n-\left(\mu_m-\mu_n\right) \right. \\ \left. \le t_2\sqrt{\frac{(m-1)S_m^2/\sigma_m^2+(n-1)S_n^2/\sigma_n^2}{m+n-2}}\sqrt{\sigma_m^2/m+\sigma_n^2/n} \right) = 1-\alpha \quad (110)$$

by suitably choosing the decision variables t_1 and t_2. Hence, the statistical confidence interval for $\bar{U}_m-\bar{Z}_n-\left(\mu_m-\mu_n\right)$ is given by

$$\left[t_1\frac{\sqrt{\frac{(m-1)S_m^2/\sigma_m^2+(n-1)S_n^2/\sigma_n^2}{m+n-2}}}{\frac{1}{\sqrt{\sigma_m^2/m+\sigma_n^2/n}}},\ t_2\frac{\sqrt{\frac{(m-1)S_m^2/\sigma_m^2+(n-1)S_n^2/\sigma_n^2}{m+n-2}}}{\frac{1}{\sqrt{\sigma_m^2/m+\sigma_n^2/n}}} \right]. \quad (111)$$

The length of the statistical confidence interval for $\bar{U}_m-\bar{Z}_n-\left(\mu_m-\mu_n\right)$ is given by

$$L\left(t_1,t_2 \mid \sqrt{\frac{(m-1)S_m^2/\sigma_m^2+(n-1)S_n^2/\sigma_n^2}{m+n-2}}\sqrt{\sigma_m^2/m+\sigma_n^2/n} \right)$$

$$=\left(t_2-t_1\right)\sqrt{\frac{(m-1)S_m^2/\sigma_m^2+(n-1)S_n^2/\sigma_n^2}{m+n-2}}\sqrt{\sigma_m^2/m+\sigma_n^2/n}. \quad (112)$$

In order to find the confidence interval of shortest-length for $\bar{U}_m-\bar{Z}_n-\left(\mu_m-\mu_n\right)$, we should find a pair of decision variables t_1 and t_2 such that (101) is minimum.

It follows from (109) and (110) that

$$\int_{t_1}^{t_2} f(t)dt = \int_0^{t_2} f(t)dt - \int_0^{t_1} f(t)dt = (1-\alpha+p)-p = 1-\alpha, \quad (113)$$

where p $(0\le p\le\alpha)$ is a decision variable,

$$\int_0^{t_2} f(t)dt = 1-\alpha+p \quad (103)$$

and

$$\int_0^{t_1} f(t)dt = p. \quad (104)$$

Then t_2 represents the $(1-\alpha+p)$ - quantile, which is given by

$$t_2 = q_{1-\alpha+p;(t(m+n-2))}, \quad (105)$$

t_1 represents the p - quantile, which is given by

$$t_1 = q_{p;(t(m+n-2))}. \quad (106)$$

The shortest length confidence interval for $\bar{U}_m-\bar{Z}_n-\left(\mu_m-\mu_n\right)$ can be found as follows:

Minimize

$$\left(t_2-t_1\right)^2 = \left(q_{1-\alpha+p;(t(m+n-2))}-q_{p;(t(m+n-2))}\right)^2 \quad (107)$$

subject to

$$0\le p\le\alpha, \quad (108)$$

The optimal numerical solution minimizing $\left(t_2-t_1\right)^2$ can be obtained using the standard computer software "Solver" of Excel 2016. If $\sigma_m^2=\sigma_n^2$, it follows from (101) that

$$L\left(t_1,t_2\,|\sqrt{\frac{(m-1)S_m^2+(n-1)S_n^2}{m+n-2}}\sqrt{\frac{m+n}{mn}}\right)=\left(t_2-t_1\right)\sqrt{\frac{(m-1)S_m^2+(n-1)S_n^2}{m+n-2}}\sqrt{\frac{m+n}{mn}}. \quad (109)$$

If, for example, m=58, n=27, $\alpha = 0.05$, $\bar{U}_m = 70.7$, $\bar{Z}_n = 76.13$, $S_m^2 = (1.8)^2$, $S_n^2 = (2.42)^2$, then the optimal numerical solution of (107) is given by

$$p = 0.025, \quad t_1 = q_{p;(t(m+n-2))} = -1.98896, \quad t_2 = q_{1-\alpha+p;(t(m+n-2))} = 1.98896 \quad (110)$$

and it follows from (99) and (109) that the 100(1$-\alpha$)% confidence interval of shortest-length (or equal tails) for $\mu_1-\mu_2$ is given by

$$\left(\mu_m-\mu_n\right)\in\left(\begin{array}{l}\left(\bar{U}_m-\bar{Z}_n\right)-t_2\sqrt{\dfrac{(m-1)S_m^2+(n-1)S_n^2}{m+n-2}}\sqrt{\dfrac{m+n}{mn}},\\ \left(\bar{U}_m-\bar{Z}_n\right)-t_1\sqrt{\dfrac{(m-1)S_m^2+(n-1)S_n^2}{m+n-2}}\sqrt{\dfrac{m+n}{mn}}\end{array}\right)=\left(-6.330947,-4.52905\right) \quad (111)$$

or

$$-6.330947 \le \mu_m-\mu_n \le -4.52905. \quad (112)$$

8. Confidence Interval for the Ratio of Means of Two Different Normal Populations

Ratio in the means is used to compare two populations of positive data. Let U_1, U_2, ..., U_m be a sample of size m from a normal population having mean μ_m and variance σ_m^2 and let U_1, ..., U_n be a sample of size n from a different normal population having mean μ_n and variance σ_n^2 and suppose that the two samples are independent of each other. We are interested in constructing a confidence interval for the ratio of means (μ_m, μ_n) of two different normal populations To obtain this confidence interval, we need the distribution of $\bar{U}_m - \kappa\bar{U}_n$, where

$$\bar{U}_m=\sum_{i=1}^{m}U_i\Big/m\sim N\left(\mu_m,\sigma_m^2/m\right), \quad \bar{U}_n=\sum_{i=1}^{n}U_i\Big/n\sim N\left(\mu_n,\sigma_n^2/n\right). \quad 113)$$

It can be shown that

$$\bar{U}_m-\kappa\bar{U}_n \sim N\left(\mu_m-\kappa\mu_n,\frac{\sigma_m^2}{m}+\frac{\kappa^2\sigma_n^2}{n}\right) \quad 114)$$

or

$$\frac{\bar{U}_m-\kappa\bar{U}_n-(\mu_m-\kappa\mu_n)}{\sqrt{\dfrac{\sigma_m^2}{m}+\dfrac{\kappa^2\sigma_n^2}{n}}}=W_1\sim N\left(0,1\right). \quad (115)$$

This is independent of

$$\sum_{i=1}^{m}\left(U_i-\bar{U}_m\right)^2\Big/\sigma_m^2=\frac{(m-1)}{\sigma_m^2}\frac{\sum_{i=1}^{m}\left(U_i-\bar{U}_m\right)^2}{(m-1)}=\frac{(m-1)S_m^2}{\sigma_m^2}\sim\chi_{m-1}^2 \quad (116)$$

and

$$\sum_{j=1}^{n}\left(U_j-\bar{U}_n\right)^2\Big/\sigma_n^2=\frac{(n-1)}{\sigma_n^2}\frac{\sum_{j=1}^{n}\left(U_j-\bar{U}_n\right)^2}{(n-1)}=\frac{(n-1)S_n^2}{\sigma_n^2}\sim\chi_{n-1}^2, \quad (117)$$

where

$$\frac{(m-1)S_m^2}{\sigma_m^2}+\frac{(n-1)S_n^2}{\sigma_n^2}=W_2\sim\chi^2(m+n-2). \quad (118)$$

It follows from (84), (115) and (118) that

$$\frac{W_1}{\sqrt{W_2/(m+n-2)}}=\frac{\bar{U}_m-\kappa\bar{U}_n-(\mu_m-\kappa\mu_n)}{\sqrt{\dfrac{\sigma_m^2}{m}+\dfrac{\kappa^2\sigma_n^2}{n}}}\frac{1}{\sqrt{\left[\dfrac{(m-1)S_m^2}{\sigma_m^2}+\dfrac{(n-1)S_n^2}{\sigma_n^2}\right]\Big/(m+n-2)}}$$

$$=\frac{\bar{U}_m-\kappa\bar{U}_n-(\mu_m-\kappa\mu_n)}{\sqrt{(m-1)S_m^2/\sigma_m^2+(n-1)S_n^2/\sigma_n^2}}\sqrt{\frac{m+n-2}{\sigma_m^2/m+\kappa^2\sigma_n^2/n}}=T(m+n-2)\sim f(t), \tag{119}$$

where $T(m+n\text{-}2)$ is a t-random variable with m + n – 2 degrees of freedom. Taking Theorem 5 into account, we have that

$$f(t)=\frac{\Gamma\left((m+n-1)/2\right)}{\sqrt{\pi(m+n-2)}\;\Gamma\left((m+n-2)/2\right)}\left[1+\frac{t^2}{m+n-2}\right]^{-(m+n-1)/2},\quad -\infty<t<\infty. \tag{120}$$

Using (119) and (120), it can be obtained a 100(1−α)% confidence interval for $\bar{U}_m-\kappa\bar{U}_n-(\mu_m-\kappa\mu_n)$ from

$$P\left(t_1\le T\left(m+n-2\,|\,\bar{U}_m-\kappa\bar{U}_n-(\mu_m-\kappa\mu_n)\right)\le t_2\right)$$

$$=P\left(t_1\le\frac{\bar{U}_m-\kappa\bar{U}_n-(\mu_m-\kappa\mu_n)}{\sqrt{(m-1)S_m^2/\sigma_m^2+(n-1)S_n^2/\sigma_n^2}}\frac{\sqrt{m+n-2}}{\sqrt{\sigma_m^2/m+\kappa^2\sigma_n^2/n}}\le t_2\right)$$

$$=P\left(\begin{array}{c}t_1\sqrt{\dfrac{(m-1)S_m^2/\sigma_m^2+(n-1)S_n^2/\sigma_n^2}{m+n-2}}\sqrt{\sigma_m^2/m+\kappa^2\sigma_n^2/n}\le\bar{U}_m-\kappa\bar{U}_n-(\mu_m-\kappa\mu_n)\\ \le t_2\sqrt{\dfrac{(m-1)S_m^2/\sigma_m^2+(n-1)S_n^2/\sigma_n^2}{m+n-2}}\sqrt{\sigma_m^2/m+\kappa^2\sigma_n^2/n}\end{array}\right)=1-\alpha \tag{121}$$

by suitably choosing the decision variables t_1 and t_2. Hence, the statistical confidence interval for $\bar{U}_m-\kappa\bar{U}_n-(\mu_m-\kappa\mu_n)$ is given by

$$\left[t_1\frac{\sqrt{\dfrac{(m-1)S_m^2/\sigma_m^2+(n-1)S_n^2/\sigma_n^2}{m+n-2}}}{\dfrac{1}{\sqrt{\sigma_m^2/m+\kappa^2\sigma_n^2/n}}},\; t_2\frac{\sqrt{\dfrac{(m-1)S_m^2/\sigma_m^2+(n-1)S_n^2/\sigma_n^2}{m+n-2}}}{\dfrac{1}{\sqrt{\sigma_m^2/m+\kappa^2\sigma_n^2/n}}}\right]. \tag{122}$$

The length of the statistical confidence interval for $\bar{U}_m-\kappa\bar{U}_n-(\mu_m-\kappa\mu_n)$ is given by

$$L\left(t_1,t_2\,|\sqrt{\frac{(m-1)S_m^2/\sigma_m^2+(n-1)S_n^2/\sigma_n^2}{m+n-2}}\sqrt{\sigma_m^2/m+\kappa^2\sigma_n^2/n}\right)$$

$$=(t_2-t_1)\sqrt{\frac{(m-1)S_m^2/\sigma_m^2+(n-1)S_n^2/\sigma_n^2}{m+n-2}}\sqrt{\sigma_m^2/m+\kappa^2\sigma_n^2/n}. \tag{123}$$

In order to find the confidence interval of shortest-length for $\bar{U}_m-\kappa\bar{U}_n-(\mu_m-\kappa\mu_n)$, we should find a pair of decision variables t_1 and t_2 such that (123) is minimum. It follows from (121) and (123) that

$$\int_{t_1}^{t_2} f(t)dt = \int_{0}^{t_2} f(t)dt - \int_{0}^{t_1} f(t)dt = (1-\alpha+p) - p = 1-\alpha, \tag{124}$$

where p $(0 \le p \le \alpha)$ is a decision variable,

$$\int_{0}^{t_2} f(t)dt = 1-\alpha+p \tag{125}$$

and

$$\int_{0}^{t_1} f(t)dt = p. \tag{126}$$

Then t_2 represents the $(1-\alpha+p)$ - quantile, which is given by

$$t_2 = q_{1-\alpha+p;(t(m+n-2))}, \tag{127}$$

t_1 represents the p - quantile, which is given by

$$t_1 = q_{p;(t(m+n-2))}. \tag{128}$$

The shortest length confidence interval for $\bar{U}_m - \kappa\bar{U}_n - (\mu_m - \kappa\mu_n)$ can be found as follows:

Minimize

$$(t_2 - t_1)^2 = \left(q_{1-\alpha+p;(t(m+n-2))} - q_{p;(t(m+n-2))}\right)^2 \tag{129}$$

subject to

$$0 \le p \le \alpha, \tag{130}$$

The optimal numerical solution minimizing $(t_2 - t_1)^2$ can be obtained using the standard computer software "Solver" of Excel 2016. If $\sigma_m^2 = \sigma_n^2$, it follows from (123) that

$$L\left(t_1, t_2 \mid \sqrt{\frac{(m-1)S_m^2 + (n-1)S_n^2}{m+n-2}}\sqrt{\frac{1}{m} + \frac{\kappa^2}{n}}\right) = (t_2 - t_1)\sqrt{\frac{(m-1)S_m^2 + (n-1)S_n^2}{m+n-2}}\sqrt{\frac{1}{m} + \frac{\kappa^2}{n}}. \tag{131}$$

If, for example, m=6, n=4, $\alpha = 0.05$, $\bar{U}_m = 117.5$, $\bar{U}_n = 126.8$, $S_m^2 = (9.7)^2$, $S_n^2 = (12)^2$, then the optimal numerical solution of (129) is given by

$$p = 0.025, \quad t_1 = q_{p;(t(m+n-2))} = -2.306, \quad t_2 = q_{1-\alpha+p;(t(m+n-2))} = 2.306 \tag{132}$$

and it follows from (121) and (131) that the 100(1−α)% confidence interval of shortest-length (or equal tails) for $\mu_1 - \kappa\mu_2$ is given by

$$\left(\begin{array}{l} \bar{U}_m - \kappa\bar{U}_n - (\mu_m - \kappa\mu_n) \ge t_1\sqrt{\frac{(m-1)S_m^2 + (n-1)S_n^2}{m+n-2}}\sqrt{\frac{1}{m} + \frac{\kappa^2}{n}}, \\ \bar{U}_m - \kappa\bar{U}_n - (\mu_m - \kappa\mu_n) \le t_2\sqrt{\frac{(m-1)S_m^2 + (n-1)S_n^2}{m+n-2}}\sqrt{\frac{1}{m} + \frac{\kappa^2}{n}} \end{array}\right) \tag{133}$$

If $\kappa = 1$, it follows from (133) that

$$(\mu_m - \mu_n) \in \left(\begin{array}{l} (\bar{U}_m - \bar{U}_n) - t_2\sqrt{\frac{(m-1)S_m^2 + (n-1)S_n^2}{m+n-2}}\sqrt{\frac{1}{m} + \frac{1}{n}}, \\ (\bar{U}_m - \bar{U}_n) - t_1\sqrt{\frac{(m-1)S_m^2 + (n-1)S_n^2}{m+n-2}}\sqrt{\frac{1}{m} + \frac{1}{n}} \end{array}\right)$$

$$=\begin{pmatrix}(117.5-126.8)-2.306\times 10.6\sqrt{\frac{1}{6}+\frac{1}{4}},\\ (117.5-126.8)+2.306\times 10.6\sqrt{\frac{1}{6}+\frac{1}{4}}\end{pmatrix}=(-25.07,\ 6.47) \tag{134}$$

or

$$-25.07<\mu_m-\mu_n<6.47. \tag{135}$$

An analytical expression for determining the optimal value of κ (the ratio in means of two different normal populations) can be obtained from (121), where it is assumed that $\sigma_m^2=\sigma_n^2$ and $(\mu_m-\kappa\mu_n)=0$:

$$\begin{pmatrix}t_1\sqrt{\frac{(m-1)S_m^2+(n-1)S_n^2}{m+n-2}}\sqrt{1/m+\kappa^2/n}\\ \le \bar{U}_m-\kappa\bar{U}_n\\ \le t_2\sqrt{\frac{(m-1)S_m^2+(n-1)S_n^2}{m+n-2}}\sqrt{1/m+\kappa^2/n}\end{pmatrix}=\begin{pmatrix}\kappa\bar{U}_n+t_1\sqrt{\frac{(m-1)S_m^2+(n-1)S_n^2}{m+n-2}}\sqrt{1/m+\kappa^2/n}\le\bar{U}_m,\\ \bar{U}_m\le\kappa\bar{U}_n+t_2\sqrt{\frac{(m-1)S_m^2+(n-1)S_n^2}{m+n-2}}\sqrt{1/m+\kappa^2/n}\end{pmatrix}$$

$$=\begin{pmatrix}\kappa+t_1\frac{\sqrt{\frac{(m-1)S_m^2+(n-1)S_n^2}{m+n-2}}}{\bar{U}_n}\sqrt{1/m+\kappa^2/n}\le\frac{\bar{U}_m}{\bar{U}_n},\\ \frac{\bar{U}_m}{\bar{U}_n}\le\kappa+t_2\sqrt{\frac{(m-1)S_m^2+(n-1)S_n^2}{m+n-2}}\sqrt{1/m+\kappa^2/n}\end{pmatrix}$$

$$=\begin{pmatrix}\kappa\le\frac{\bar{U}_m}{\bar{U}_n}-t_1\frac{\sqrt{\frac{(m-1)S_m^2+(n-1)S_n^2}{m+n-2}}}{\bar{U}_n}\sqrt{1/m+\kappa^2/n},\\ \kappa\ge\frac{\bar{U}_m}{\bar{U}_n}-t_2\frac{\sqrt{\frac{(m-1)S_m^2+(n-1)S_n^2}{m+n-2}}}{\bar{U}_n}\sqrt{1/m+\kappa^2/n}\end{pmatrix}$$

$$=\begin{pmatrix}\kappa\le 0.926656+2.306\frac{10.6}{126.8}\sqrt{1/6+\kappa^2/4},\\ \kappa\ge 0.926656-2.306\frac{10.6}{126.8}\sqrt{1/6+\kappa^2/4}\end{pmatrix}=\begin{pmatrix}\kappa\le 0.926656+0.192773\sqrt{0.166667+0.25\kappa^2},\\ \kappa\ge 0.926656-0.192773\sqrt{0.166667+0.25\kappa^2}\end{pmatrix}$$

$$\Rightarrow\begin{pmatrix}\text{minimize:}\\ \left(\kappa-0.926656-0.192773\sqrt{0.166667+0.25\kappa^2}\right)^2,\\ \left(\kappa-0.926656+0.192773\sqrt{0.166667+0.25\kappa^2}\right)^2,\\ \text{subject to: } \kappa\ge 0.\end{pmatrix}=\begin{pmatrix}\kappa\le 1.05526,\\ \kappa\ge 0.815431\end{pmatrix}. \tag{136}$$

Thus, it follows from (136) that

$$\kappa\in(0.815431,\ 1.05526). \tag{137}$$

9 Conclusion

The new intelligent computational models proposed in this paper are conceptually simple, efficient, and useful for constructing accurate statistical tolerance or prediction limits and shortest-length or equal-tailed confidence intervals under the parametric uncertainty of applied stochastic models. The methods listed above are based on adequate computational models of the cumulative distribution function of order statistics and constructive use of the invariance principle in mathematical statistics. These methods can be used to solve real-life problems in all areas including engineering, science, industry, automation & robotics, machine learning, business & finance, medicine and biomedicine, optimization, planning and scheduling.

Conflicts of Interest

The authors declare that they have no conflict of interest.

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